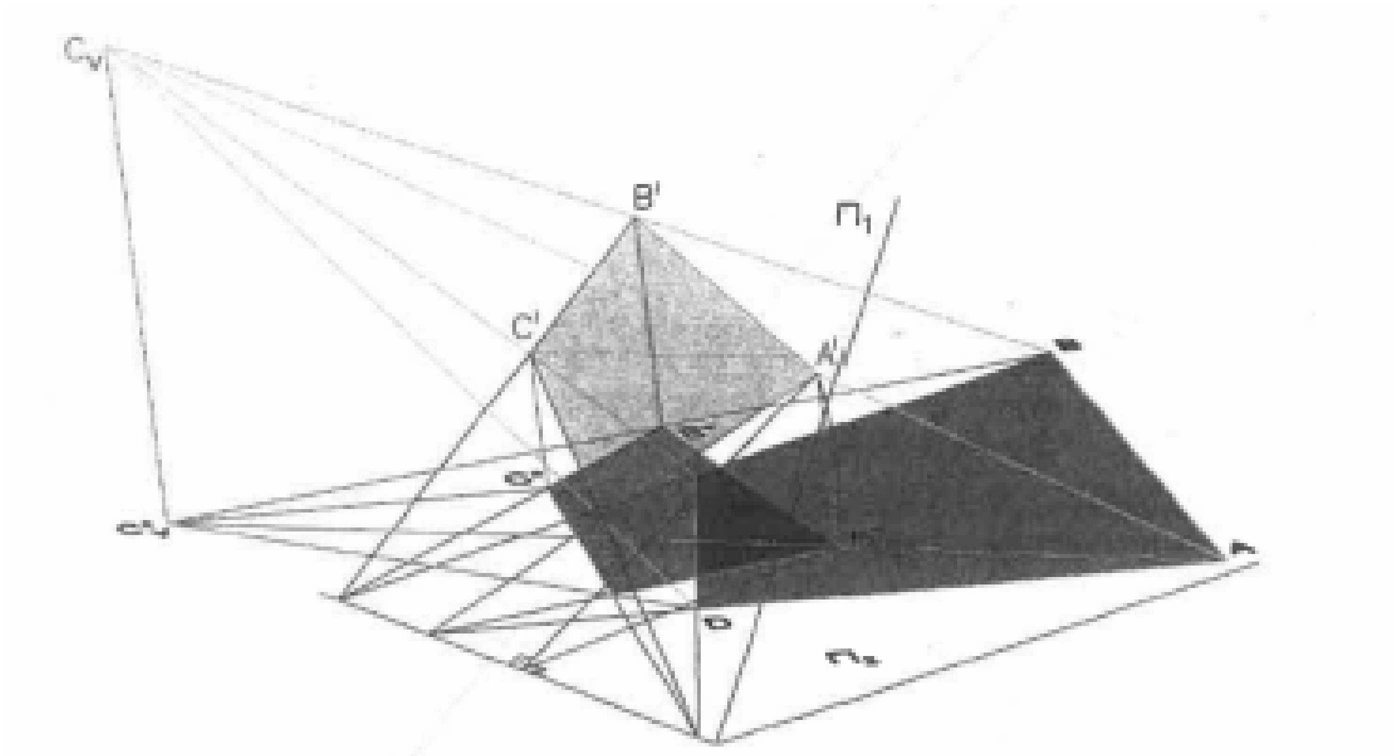
Mathematics for Architects I - Basis of visualization



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Pécs

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Chapter 1

Vectors

As a first approach, we can say that a vector is a directed line segment, that is determined by its

- magnitude and
- direction.

  Directed line segments with the same direction and magnitude represent the same vector.

Notation: $\underline{a}$, $\underline{b}$, $\underline{c}$, ... ($\bar{a}$, $\bar{b}$, $\bar{c}$, ...) or $\overline{AB}$.

1. Basic concepts regarding vectors.

   (a) Parallel vectors.

   (b) Vectors being parallel and having the same direction.

   (c) Equal vectors: if they are parallel, they have the same direction their magnitudes are equal as well.
   Notation: $\underline{a} = \underline{b}$,

   (d) Absolute value of a vector
   The absolute value of a vector $\underline{v}$ is its magnitude, denoted by $|\underline{v}|$.

   (e) Zero vector
   A vector with magnitude zero (whose starting point is identical to its endpoint) is called a zero vector, or null vector.
   Notation: $\underline{0}$.
   The direction of the zero vector is arbitrary.

   (f) Unit vector
   A vector with magnitude 1 (having arbitrary direction) is called a unit vector.

(g) The angle between two vectors

The angle between two vectors is defined as the angle between two half lines, starting from a single point, having the same direction as the given vectors. It can not be bigger than 180 degree by definition, and it can not be negative. It does not depend on the order of the vectors.

2. Vector operations

(a) Addition of vectors

- Parallelogram rule
  Let $\underline{a}$ and $\underline{b}$ be two non-parallel vectors. Consider a representation for both vectors that are starting from the same point A. Then from the endpoint of the vector $\underline{a}$ draw a line parallel to $\underline{b}$, and vice versa. This way we get a parallelogram, and the vector represented by the diagonal, starting at A, is the *sum of the two vectors*, $\underline{a} + \underline{b}$, same as $\underline{b} + \underline{a}$.
- Polygon rule
  Let the vectors $\underline{a}$ and $\underline{b}$ given (with a representation given graphically). Translate the vector $\underline{b}$ so that its starting point will be the same as the endpoint of the vector $\underline{a}$. The vector starting from the starting point of a and ending in the endpoint of b is their sum, the vector $\underline{a} + \underline{b}$.
  The triangle inequality ensures that $|\underline{a} + \underline{b}| \leq |\underline{a}| + |\underline{b}|$.
- Properties of the addition of vectors:
  (i) $\underline{a} + \underline{b} = \underline{b} + \underline{a}$ - commutativity;
  (ii) $\underline{a} + (\underline{b} + \underline{c}) = (\underline{a} + \underline{b}) + \underline{c}$ - associativity;
  (iii) $\underline{a} + \underline{0} = \underline{a}$.

(b) Subtraction of vectors.

The difference of the vectors $\underline{a}$ and $\underline{b}$, denoted by $\underline{a} - \underline{b}$, is a vector that added to the vector $\underline{b}$ results in the vector $\underline{a}$ (as their sum).

The difference of the vectors $\underline{a}$ and $\underline{b}$ can be represented graphically so that you translate $\underline{a}$ and $\underline{b}$ to a common starting point, and $\underline{a} - \underline{b}$ starts from the endpoint of the vector $\underline{b}$, pointing to the endpoint of the vector $\underline{a}$.

(c) Multiplication of a vector by a real number (a scalar).

For $\lambda \in \mathbb{R}$ the vector $\lambda\underline{a}$ is a vector

- whose magnitude is $|\lambda\underline{a}| = |\lambda||\underline{a}|$;
- parallel to $\underline{a}$;
- its direction is
  - the same as the direction of $\underline{a}$ if $\lambda > 0$,
  - reverse of the direction of $\underline{a}$ if $\lambda < 0$,
  - arbitrary if $\lambda = 0$.

Identities regarding the scalar multiplication of a vector:

- $(\lambda\mu)\underline{a} = \lambda(\mu\underline{a})$
- $(\lambda + \mu)\underline{a} = (\lambda\underline{a} + \mu\underline{a})$
- $\lambda(\underline{a} + \underline{b}) = \lambda\underline{a} + \lambda\underline{b}$ where $\lambda,\ \mu \in \mathbb{R}$.

3. Linear combination of vectors

   A linear combination of the vectors $\underline{a}_1, \underline{a}_2, ..., \underline{a}_n$ $(n \in \mathbb{N}^+)$ is a vector $\underline{a} = \lambda_1\underline{a}_1 + \lambda_2\underline{a}_2 + ... + \lambda_n\underline{a}_n$ $(\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{R})$.

   (a) The system of vectors $\underline{a}_1, \underline{a}_2, ..., \underline{a}_n$ is said to be linearly independent if $\lambda_1\underline{a}_1 + \lambda_2\underline{a}_2 + ... + \lambda_n\underline{a}_n = \underline{0}$ is only fulfilled with the values $\lambda_1 = \lambda_2 = ... = \lambda_n = 0$.

   (b) If the previous equality can also be fulfilled so that at least one of the λ's is not zero, then the given system of vectors is said to be linearly dependent.

   We call $0\underline{a}_1 + 0\underline{a}_2 + ... + 0\underline{a}_n = \underline{0}$ the trivial linear combination yielding the zero vector.

Given three non-coplanar vectors in the (3-dimensional) space (non-coplanar means that they are not in the same plane) , any vector in the space can be produced as their linear combination, and this linear combination is unique. E.g. $\underline{v} = 0.7\underline{a} + 1.2\underline{b} + 1.3\underline{c}$ (if $\underline{a}, \underline{b}, \underline{c}$ are not coplanar) Then we can use the notation $\underline{v} = [0.7,\ 1.2,\ 1.3]$ to identify the vector $\underline{v}$.

   (c) 3 linearly independent vectors of the space can be called a *basis* of the space.

   (Equivalently, three vectors of the space that are not in the same plane can also be called a basis, – since they are linearly independent.)

   Based on the above, any vector $\underline{v}$ in the space can be generated in the form $\underline{v} = \alpha\underline{a} + \beta\underline{b} + \gamma\underline{c}$ $(\alpha, \beta, \gamma \in \mathbb{R})$ uniquely in the space defined (spanned) by the vectors $\underline{a}, \underline{b}, \underline{c}$.

4. Coordinates of vectors.

   If $\underline{a}, \underline{b}, \underline{c}$ is a basis of the space, then for any $\underline{v}$ vector in the space, the coefficients α, β, γ in the linear combination $\underline{v} = \alpha\underline{a} + \beta\underline{b} + \gamma\underline{c}$ are called the *coordinates* of $\underline{v}$ with respect to the basis $\underline{a}, \underline{b}, \underline{c}$.

   Notation: $\underline{v} = [\alpha, \beta, \gamma]$.

   If $\underline{i}, \underline{j}, \underline{k}$ are unit vectors and they are perpendicular to each other, we call it a *standard basis*. (By convention, the elements of the standard basis form a right-handed system in this order: this is one of the possible two orientations; you can imagine it so that the thumb on your right hand points in the $\underline{i}$ direction, the first finger in the $\underline{j}$ direction, and the middle finger in the $\underline{k}$ direction.)

Let v_1, v_2, v_3 denote the coordinates of the vector $\underline{v}$ with respect to the standard basis ($\underline{v} = v_1\underline{i} + v_2\underline{j} + v_3\underline{k}$).

(a) Two vectors given by their coordinates are equal if and only if their corresponding coordinates are equal.

(b) The coordinates of $\lambda\underline{v}$ are λ-times the coordinates of $\underline{v}$:
$\lambda\underline{v} = (\lambda v_1)\underline{i} + (\lambda v_2)\underline{j} + (\lambda v_3)\underline{k} \qquad (\lambda \in \mathbb{R})$.

(c) If $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$, then
$\underline{a} \pm \underline{b} = (a_1 \pm b_1)\underline{i} + (a_2 \pm b_2)\underline{j} + (a_3 \pm b_3)\underline{k}$.

5. Multiplication of vectors

(a) *Scalar product* (or "dot product"), resulting in a scalar (a real number):

The scalar product of the vectors $\underline{a}$ and $\underline{b}$ is defined as $\underline{a}.\underline{b} = |\underline{a}||\underline{b}|\cos\phi$, where ϕ is the angle enclosed by $\underline{a}$ and $\underline{b}$.

The most important properties of the scalar product:

- $\underline{a}.\underline{b} = \underline{b}.\underline{a}$ (commutativity);
- $\underline{a}.(\underline{b} + \underline{c}) = \underline{a}.\underline{b} + \underline{a}.\underline{c}$ (distributivity);
- $\lambda(\underline{a}.\underline{b}) = (\lambda\underline{a}).\underline{b} = \underline{a}.(\lambda\underline{b})$;
- $\underline{a}.(\underline{b}.\underline{c}) \neq (\underline{a}.\underline{b}).\underline{c}$ because the product on the left hand side is a scalar multiple of $\underline{a}$, so it is parallel to $\underline{a}$, and the product on the right hand side is a scalar multiple of $\underline{c}$, so it is parallel to $\underline{c}$. This means they are different vectors unless $\underline{a}$ is parallel to $\underline{c}$. (So the scalar product is not associative.)

Theorem: If $\underline{a}$ and $\underline{b}$ are given with their coordinates $[a_1, a_2, a_3]$ and $[b_1, b_2, b_3]$ in the standard basis, their scalar product can be calculated by adding the product of the corresponding coordinates of the vectors:

$$\underline{a}.\underline{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

Proof: $\underline{a}.\underline{b} = (a_1\underline{i}+a_2\underline{j}+a_3\underline{k}).(b_1\underline{i}+b_2\underline{j}+b_3\underline{k}) = a_1b_1\underline{i}.\underline{i}+a_1b_2\underline{i}.\underline{j}+a_1b_3\underline{i}.\underline{k}+a_2b_1\underline{j}.\underline{i}+a_2b_2\underline{j}.\underline{j}+a_2b_3\underline{j}.\underline{k}+a_3b_1\underline{k}.\underline{i}+a_3b_2\underline{k}.\underline{j}+a_3b_3\underline{k}.\underline{k} = a_1b_1+a_2b_2+a_3b_3$, because $\underline{i}.\underline{i} = |\underline{i}||\underline{i}|\cos 0° = 1, ...$; $\underline{i}.\underline{j} = |\underline{i}||\underline{j}|\cos 90° = 0$, and similarly, $\underline{j}.\underline{j} = \underline{k}.\underline{k} = 1$ and $\underline{i}.\underline{k} = \underline{j}.\underline{k} = 0$.

(b) *Vectorial product* (or "cross product"), resulting in a vector:

The vectorial product $\underline{a}\times\underline{b}$ of the vectors $\underline{a}$ and $\underline{b}$ is defined as follows:

- its magnitude is $|\underline{a}\times\underline{b}| = |\underline{a}||\underline{b}|\sin\phi$, where ϕ is the angle enclosed by $\underline{a}$ and $\underline{b}$;
- its direction is perpendicular to both $\underline{a}$ and $\underline{b}$;
- $\underline{a}$, $\underline{b}$ and $\underline{a} \times \underline{b}$ form a right-handed system in this order.

The most important properties of the vectorial product:

- $\underline{a} \times \underline{b} \neq \underline{b} \times \underline{a}$ (it is not commutative), but
- $\underline{a} \times \underline{b} = -(\underline{b} \times \underline{a})$;
- $\underline{a} \times (\underline{b} + \underline{c}) = \underline{a} \times \underline{b} + \underline{a} \times \underline{c}$ and
- $(\underline{b} + \underline{c}) \times \underline{a} = \underline{b} \times \underline{a} + \underline{c} \times \underline{a}$ (distributivity);
- $\lambda(\underline{a} \times \underline{b}) = (\lambda\underline{a}) \times \underline{b} = \underline{a} \times (\lambda\underline{b})$ for any $\lambda \in \mathbb{R}$;
- $\underline{a} \times (\underline{b} \times \underline{c}) \neq (\underline{a} \times \underline{b}) \times \underline{c}$ (it is not associative, e.g. $\underline{i} \times (\underline{i} \times \underline{j}) = \underline{i} \times \underline{k} = -\underline{j} \neq (\underline{i} \times \underline{i}) \times \underline{j} = \underline{0} \times \underline{j} = \underline{0}$), but
- $\underline{a}.(\underline{b} \times \underline{c}) = (\underline{a} \times \underline{b}).\underline{c}$ – this is called the triple product of the vectors $\underline{a}, \underline{b}, \underline{c}$, yielding the volume of the parallelepiped spanned by them.

Theorem: If $\underline{a}$ and $\underline{b}$ are given with their coordinates $[a_1, a_2, a_3]$ and $[b_1, b_2, b_3]$ in the standard basis, their vectorial product can be calculated as follows:

$$\underline{a} \times \underline{b} = (a_2b_3 - a_3b_2)\underline{i} - (a_1b_3 - a_3b_1)\underline{j} + (a_1b_2 - a_2b_1)\underline{k}$$

$$= \big[(a_2b_3 - a_3b_2),\ -(a_1b_3 - a_3b_1),\ (a_1b_2 - a_2b_1)\big].$$

Proof: $\underline{a} \times \underline{b} = (a_1\underline{i} + a_2\underline{j} + a_3\underline{k}) \times (b_1\underline{i} + b_2\underline{j} + b_3\underline{k}) = a_1b_1(\underline{i} \times \underline{i}) + a_1b_2(\underline{i} \times \underline{j}) + a_1b_3(\underline{i} \times \underline{k}) + a_2b_1(\underline{j} \times \underline{i}) + a_2b_2(\underline{j} \times \underline{j}) + a_2b_3(\underline{j} \times \underline{k}) + a_3b_1(\underline{k} \times \underline{i}) + a_3b_2(\underline{k} \times \underline{j}) + a_3b_3(\underline{k} \times \underline{k}) = (a_2b_3 - a_3b_2)\underline{i} - (a_1b_3 - a_3b_1)\underline{j} + (a_1b_2 - a_2b_1)\underline{k}$, because $\underline{i} \times \underline{i} = \underline{j} \times \underline{j} = \underline{k} \times \underline{k} = \underline{0}$; $\underline{i} \times \underline{j} = \underline{k}$, $\underline{i} \times \underline{k} = -\underline{j}$, $\underline{j} \times \underline{i} = -\underline{k}$, $\underline{j} \times \underline{k} = \underline{i}$, $\underline{k} \times \underline{i} = \underline{j}$, $\underline{k} \times \underline{j} = -\underline{i}$.

An other (easier to remember) form for the calculation the vectorial product can be found at the end of the next chapter.

Chapter 2

Matrices

Definition: A matrix (plural: matrices) is a set of elements, where the elements have fixed positions, – arranged in rows and columns. An $n \times m$ matrix has nm elements in a rectangular form: in n rows and m columns.

The elements of a matrix can be real numbers or complex numbers, functions, vectors or even matrices. In the further text – if nothing else is stated – the elements of the matrices will be real numbers.

It is useful to consider the matrix as one mathematical object and denote it by a singular letter. In print we use boldface latin capital letters, in handwriting we usually use capitals underlined twice.

$$\boxed{\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \dots & \dots & & \dots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix} = [a_{ij}] \ .}$$

If we want to emphasise that the matrix has n rows and m columns, we can write the following:

$$\mathbf{A}_{n\times m} = [a_{ij}]_{nm} \ .$$

Definition: Two matrices are equal if they are equal in size (number of rows and number of columns), and all their elements standing in the corresponding positions are equal. So the matrix equation

$$\mathbf{A}_{n\times m} = \mathbf{B}_{n\times m}$$

means nm simple equations.

For the matrices

$$\mathbf{A}_{3\times 2} = \begin{bmatrix} 2 & 3 \\ 1 & -2 \\ 4 & 0 \end{bmatrix}, \quad \mathbf{B}_{2\times 3} = \begin{bmatrix} 2 & 1 & 4 \\ 3 & -2 & 0 \end{bmatrix}, \quad \mathbf{C}_{2\times 3} = \begin{bmatrix} 2 & 1 & 4 \\ 3 & -2 & 8 \end{bmatrix}$$

$\mathbf{A} \neq \mathbf{B}$ because they are not of the same size, and $\mathbf{B} \neq \mathbf{C}$ because they are of the same size but their a_{23} elements are different.

The operation of addition and subtraction can only be defined for matrices of the same size.

Definition: The sum or product of the following two matrices (whose elements are real numbers)

$$\mathbf{A}_{n\times m} = [a_{ij}] \quad \text{and} \quad \mathbf{B}_{n\times m} = [b_{ij}]$$

is defined to be the matrix

$$\mathbf{C}_{n\times m} = [c_{ij}]$$

whose elements are

$$c_{ij} = a_{ij} \pm b_{ij} \qquad (i = 1, ..., n; \;\; j = 1, ..., m)\,.$$

The addition and subtraction of matrices is commutative and associative:

$$\begin{gathered} \mathbf{A}+\mathbf{B} = \mathbf{B}+\mathbf{A} \\ (\mathbf{A}+\mathbf{B})+\mathbf{C} = \mathbf{A}+(\mathbf{B}+\mathbf{C})\,. \end{gathered}$$

Definition: Consider a matrix $\mathbf{A} = [a_{ij}]$ and a scalar $\lambda \in \mathbb{R}$. The matrix $\mathbf{B} = \lambda\mathbf{A}$ has the same size as $\mathbf{A}$, and its elements are

$$\boxed{b_{ij} = \lambda a_{ij}.}$$

This means is we multiply a matrix by a real number, we have to multiply each element of the matrix by the given number.

The multiplication of a matrix with a scalar is a *commutative, associative* and *distributive* operation:

$$\begin{gathered} (\lambda\mu)\mathbf{A} = \lambda(\mu\mathbf{A}) = \mu(\lambda\mathbf{A}) \\ (\lambda+\mu)\mathbf{A} = \lambda\mathbf{A}+\mu\mathbf{A} \quad \text{and} \quad \lambda(\mathbf{A}+\mathbf{B}) = \lambda\mathbf{A}+\lambda\mathbf{B}\,. \end{gathered}$$

For $(-1)\mathbf{A}$ we usually write $-\mathbf{A}$.

The *product* $\mathbf{AB}$ of the matrix $\mathbf{A}$ and the matrix $\mathbf{B}$ is only defined if they fit in size: the number of columns of the left factor ($\mathbf{A}$) must be equal to the number of rows of the right factor ($\mathbf{B}$). In this case we say that $\mathbf{A}$ and $\mathbf{B}$ are conformable for multiplication (in this order). The matrices $\mathbf{A}_{n\times m}$ and $\mathbf{B}_{m\times p}$ are conformable for multiplication in the sequence $\mathbf{A}$, $\mathbf{B}$; but they are not conformable in the sequence $\mathbf{B}$, $\mathbf{A}$ if $n \neq p$.

Definition. The product of the matrices $\mathbf{A} = [a_{ij}]$ of size $n \times m$ and $\mathbf{B} = [b_{jk}]$ of size $m \times p$ is the matrix $\mathbf{C} = [c_{ik}]$ of size $n \times p$ such that

$$\boxed{c_{ik} = \sum_{j=1}^{m} a_{ij}b_{jk} = a_{i1}b_{1k} + a_{i2}b_{2k} + ... + a_{im}b_{mk}}$$

for all $i = 1, ..., n$ and $k = 1, ..., p$.

As an example for matrix multiplication, let's multiply the following two matrices.

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ -1 & 1 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 4 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}.$$

These are conformable in the sequence $\mathbf{A}$, $\mathbf{B}$; their product is

$$\mathbf{AB} = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ -1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 4 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 \cdot 4 + 3 \cdot 0 & 1 \cdot 0 + 3 \cdot (-1) & 1 \cdot (-1) + 3 \cdot 0 \\ 2 \cdot 4 + 0 \cdot 0 & 2 \cdot 0 + 0 \cdot (-1) & 2 \cdot (-1) + 0 \cdot 0 \\ (-1) \cdot 4 + 1 \cdot 0 & (-1) \cdot 0 + 1 \cdot (-1) & (-1) \cdot (-1) + 1 \cdot 0 \\ 0 \cdot 4 + 2 \cdot 0 & 0 \cdot 0 + 2 \cdot (-1) & 0 \cdot (-1) + 2 \cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -3 & -1 \\ 8 & 0 & -2 \\ -4 & -1 & 1 \\ 0 & -2 & 0 \end{bmatrix}.$$

It is easy to make a mistake at the calculation of the product matrix. For this reason, it is useful to put the matrices to be multiplied in such a way that you won't spoil the position of the calculated element. The following setting shows this possibility, that is called *Falk's scheme*.

$$\begin{array}{rcc} & \mathbf{B} = & \begin{bmatrix} 4 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \\ \mathbf{A} = & \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ -1 & 1 \\ 0 & 2 \end{bmatrix} & \begin{bmatrix} 4 & -3 & -1 \\ 8 & 0 & -2 \\ -4 & -1 & 1 \\ 0 & -2 & 0 \end{bmatrix} = \mathbf{AB}. \end{array}$$

The matrix with only zero elements is called a zero matrix. E.g.

$$\mathbf{0}_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is a zero matrix.

The identity matrix, or sometimes called a unit matrix is a square matrix with ones on the main diagonal and zeros elsewhere. It is denoted by I_n, or simply by I if the size is irrelevant or can be trivially determined by the context. E.g.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is an identity matrix.

Properties of the matrix multiplication:

$$\mathbf{AB} \neq \mathbf{BA}$$

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$$

$$(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC}+\mathbf{BC} \quad \text{and} \quad \mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}\,.$$

If there is a scalar factor in a product, the product does not depend on the position of the scalar:

$$(\lambda\mathbf{A})\mathbf{B} = \mathbf{A}(\lambda\mathbf{B}) = \lambda(\mathbf{AB})\,.$$

The multiplication of matrices can not be commutative in general, because the factors may not even be conformable for multiplication after interchanging them. There is no conformability problem in the case of equal size square matrices, but commutativity still only rarely holds. Some examples, though:

$$\mathbf{0A} = \mathbf{A0} = \mathbf{0}$$

$$\mathbf{IA} = \mathbf{AI} = \mathbf{A}\,.$$

2.1 The determinant of a matrix

The *determinant* of a matrix is a very useful concept. It is defined recursively: For a 2×2 matrix

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$

and for an $n \times n$ matrix ($n > 2$), its determinant is defined as follows.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{vmatrix} =$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} & \dots & a_{2n} \\ a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots \\ a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n3} & \dots & a_{nn} \end{vmatrix} \pm \dots + (-1)^{(n-1)} a_{1n} \begin{vmatrix} a_{21} & a_{22} & \dots & a_{2,n-1} \\ a_{31} & a_{32} & \dots & a_{3,n-1} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{n,n-1} \end{vmatrix},$$

where the signs of the consecutive terms are alternating.

E.g. for $n = 3$:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Using this definition,

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

where $\underline{a} = (a_1, a_2, a_3)$ and $\underline{b} = (b_1, b_2, b_3)$. (See the definition and calculation of the vectorial product at the end of the previous chapter.)

Chapter 3

Geometric planning

A major part of technical design is the description of the spacial properties of the objects. The creation of plans and drawings - that are some specific geometric status descriptions - can be made based on mappings that preserve all the properties of the object, and which must also be practical. (The elements of a geometric description can be line segments − intersections with planes −, axonometric and perspective images, or graphs that show the relationships properly).

Planning means more than geometric planning. The geometric type of information is just a part of the total set of information.

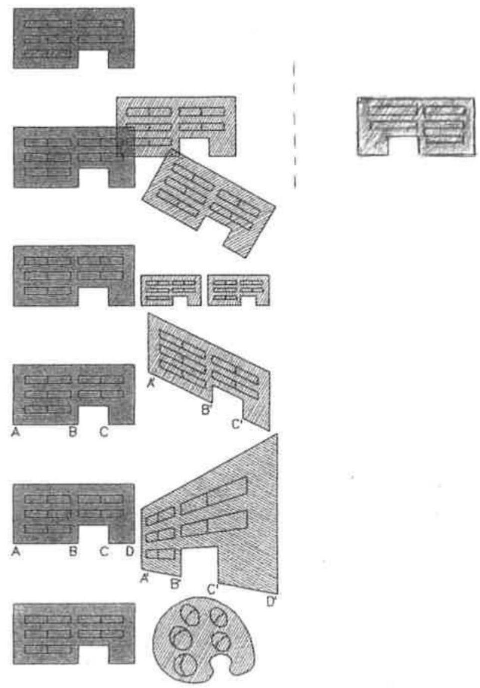
There are different mathematical − geometric − systems (spaces) available for solving geometric and visual problems. It is an essential question how to choose those geometric systems that are useful for the description of the shape of buildings.

3.1 Geometric mappings in the plane applied at architectural planning

We can use numbers for both the qualitative and quantitative description of objects in nature. Multiple different branches of mathematics deal with mappings. When the mappings use graphical elements, then the study of these mappings belongs to the field of geometry. Architects and other construction planners apply many different kinds of geometric mappings. We will try to cover and describe these in a systematic way.

Consider some commonly used mappings or transformations from a plane to itself or to another plane:

- If we copy the image of an object in the plane to another position in the plane by translating it, rotating it or reflecting it to a straight line, or changing the scale of the image, then we are speaking about congruence and similarity transformations in the plane (see images 3.1.1. - 3.1.3.).



3.1.1

3.1.2

3.1.3

3.1.4

3.1.5

3.1.6

- If we project the image of an object in the plane to another plane along *parallel* rays, then the angles in the image usually change after the projection, but the image of parallel lines will still be parallel. We call the result of such a transformation the *affine*[1] analogue of the original image.

  If we identify the plane containing the original image with the new plane (the "image plane"), we can speak about the mapping of the plane to itself.

  It can be proven that these transformations preserve the *ratio* of parallel line segments, or more accurately, they bring any two segments of a straight line to two segments of another straight line having the same proportion. (See image 3.1.4: an example is $\overline{AB} : \overline{BC} = \overline{A'B'} : \overline{B'C'}$.)

  The above described transformations called parallel projections constitute the group of affine transformations, including the group of similarity transformations as a subgroup.

- Image 3.1.5 shows the *perspective* image of a building.

  You can think about perspective transformations of a plane Π so that you connect the points of the plane with a standpoint S outside the plane with straight lines, that will intersect an other plane Π' in the images of the original points.

[1] Affinity means relatives in Latin. In the old Roman Law it also means brother-in-law and sister-in-law.

Then you can identify the points of Π' with the points of Π.

In the image gained this way, the images of straight lines will still be straight lines, but it does not necessarily keep the images of parallel lines parallel. The angles and the distances in the image can change. The perspective transformations will not keep the ratio of parallel line segments in general, but it will preserve the "ratio of ratios" – the *cross ratio* (see image 3.1.5):

$$\frac{\overline{AC}}{\overline{CB}} \div \frac{\overline{AD}}{\overline{DB}} = \frac{\overline{A'C'}}{\overline{C'B'}} \div \frac{\overline{A'D'}}{\overline{D'B'}}.$$

Again, the group of the projective transformations contains the group of all previously described transformations: the affine transformations as a subgroup.

- The next level of transformations is that of topological transformations. Only the connections (lines, but not necessary straight lines) between the points, and the intersections of lines are kept through these transformations. You can see an example in image 3.1.6.

3.2 Distinguishing geometries of the plane

The question is, what organizing principle to apply when we want to distinguish different types of geometries? According to the famous program of Erlangen (the inaugural speech of Felix Klein in the University of Erlangen in 1872) the criterion for distinguishing theorems belonging to different systems of geometry is the group of transformations that keep the validity (the invariance) of the given theorem.

3.2.1 Euclidean geometry

Let us consider the properties of the similarity transformations in the plane. The angles remain unchanged at similarity transformations (translation, rotation, magnification, contraction and reflection), but the distances may change. The proportion of *any* two line segments remain the same after these transformations, even if they were not on the same straight line. (See images 3.1.1 - 3.1.3.)

The system of properties (theorems) unchanged by the group of similarity transformations, that is, the invariants of the similarity transformation group, is called Euclidean geometry.

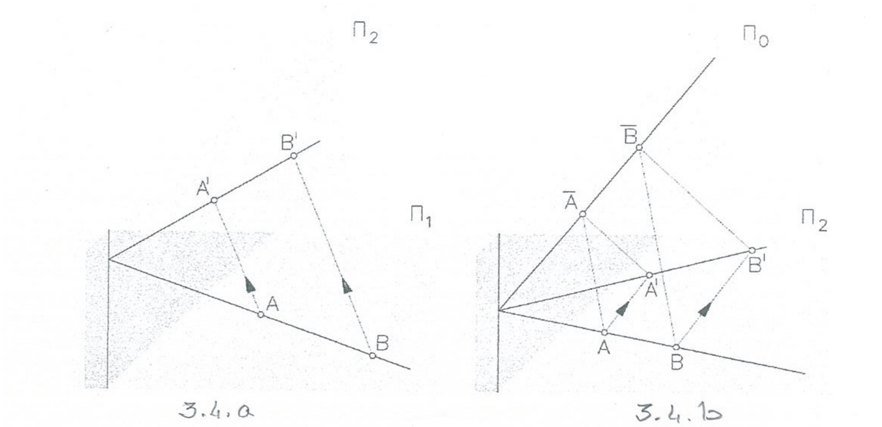
In the geometry of the plane, we can define different figures (triangles, quadrangles, circles, ellipses, etc.). We say that two figures are *equivalent*[a] *within a geometric system* if the group of transformations defining the system has at least one element mapping the first figure to the second figure.

[a]Equal in value, function, meaning, etc.

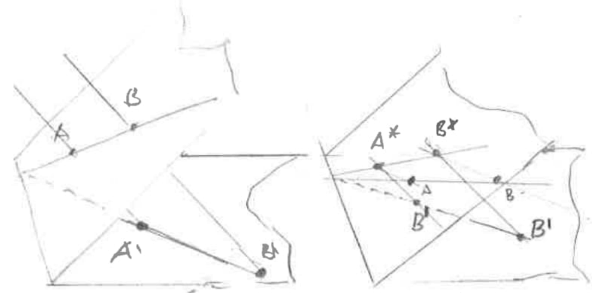
Any geometry means the set of concepts (properties) and theorems that are invariant (do not change) through the application of mappings belonging to a well-defined group of transformations.

3.3 Affine geometry

Consider a mapping of the points of a plane to another (or identical) plane that can be derived by the consecutive application of one or more parallel projections. (See image 3.4.)



3.4. fig.



3.4.1.a 3.4.1.b

3.4.1

The image 3.4.a) shows a parallel projection of the plane Π_1 to the plane Π_2, that is uniquely defined by the point A and its image A'. (AB is mapped to $A'B'$.)

The image 3.4.b) shows the projection of Π_1 to itself through two parallel projections, to and back from the plane Π_2, defined by the point A, its image $\overline{A}$, and A', the image of $\overline{A}$. (At the end, AB is mapped to $A'B'$.)

If we rotate the plane Π_2 into the plane Π_1 along the line of intersection (which is actually equivalent to a parallel projection), then the $AB \to A'B'$ mapping in image 3.4.a) can also be understood as a mapping of Π_1 to itself.

It is easy to understand the parallel projections from one plane to another plane, and the consecutive application of transformations belonging to this kind may yield transformations within the original plane, which are not so obviously defined directly (within the plane). However, let us call these transformations in the plane parallel projections as well.

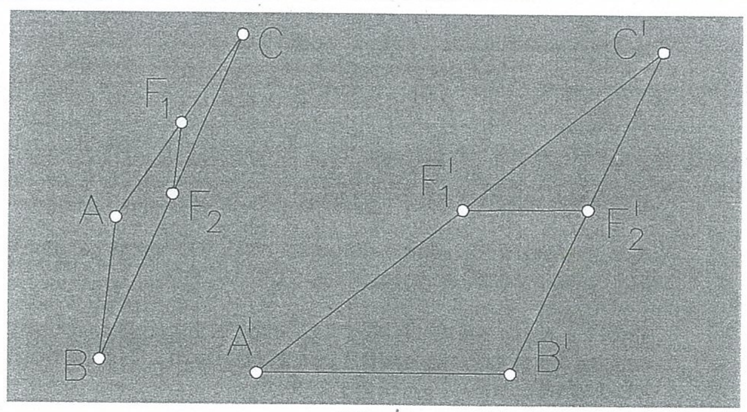
Let us add the parallel projections (either in the plane or in the space) as elements to the group of the similarity transformations already discussed, and consider the generated group of transformations (gaining new elements by the consecutive application of any sequence of the elements already defined). This will be called the group of *affine transformations*.

The affine geometry of the plane is defined by the group of affine transformations in the plane; its subject is to study the properties that are invariant under affine transformations.

Some important properties of the affine transformations:

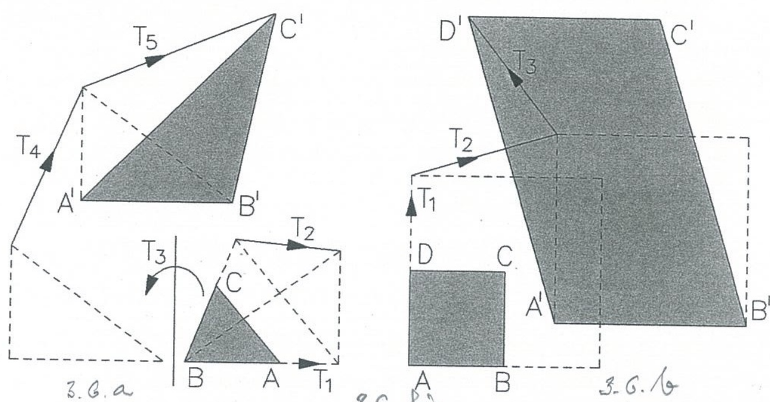
1. Under the affine transformations, the image of a point is a point, the image of a straight line is a straight line, and the images of parallel lines are parallel lines.

2. Under the affine transformations, the size of the line segments and the angles usually change, the ratio of the length of non-parallel line segments also changes, but it preserves the ratio of the length of parallel line segments. Consequently, the midpoint of a line segment is also mapped to the midpoint of the image. (See image 3.5.)



3.5 fig.

3. If two figures are equivalent in the euclidean geometry, they are necessarily equivalent in the affine geometry, too. The reverse is not always true. Two triangles are always equivalent in the affine geometry, but not in the euclidean. A square is always equivalent to any parallelogram in the affine geometry, because there always exists an affine transformation mapping the former to the latter (see Image 3.6.), but of course they are not equivalent in the Euclidean geometry.



3.6. fig

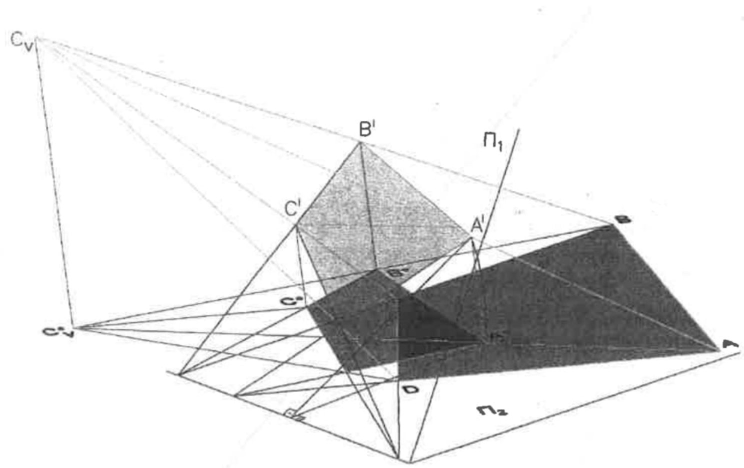
Image 3.6.a) T_1 =magnification, T_2 =parallel projection, T_3 =reflection, T_4 =translation, T_5 =parallel projection.

Image 3.6.b) T_1 =magnification, T_2 =translation, T_3 =parallel projection.

> The affine geometry is the generalization of the euclidean geometry, because the similarity transformations form a subset (subgroup) of the group of affine transformations.

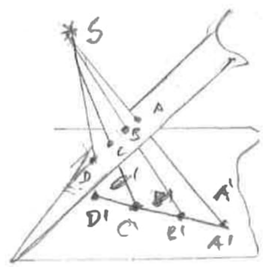
3.4 Projective geometry of the plane

The affine geometry can be generalized further by allowing *central projections*, apart prom projections along parallel rays and similarity transformations. We consider two figures identical if and only if they can be transformed to each other using this group of transformations. E.g. the figures $ABCD$ and $A'B'C'D'$ in the image 3.5.1 can be considered identical.



Π_1 sík Π_2 síkra való centrális (azaz perspektív) leképezése

3.5.1

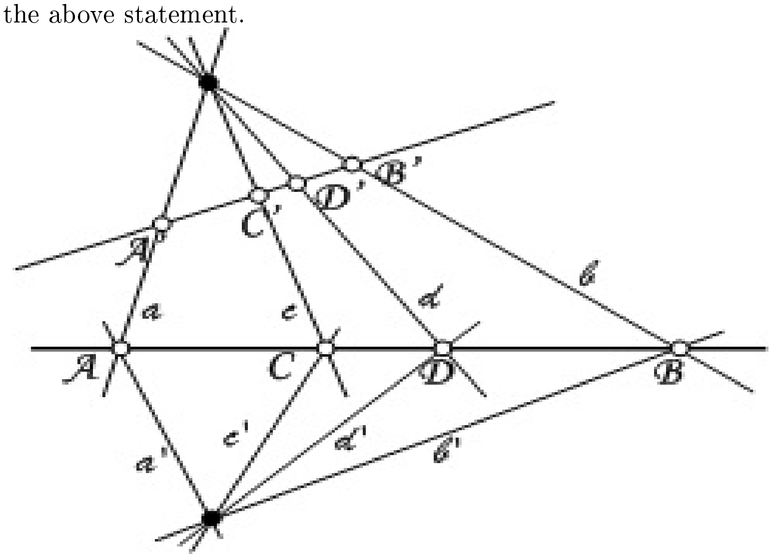


3.5.1

If Π_1 and Π_2 are two intersecting planes, and S is a point (our standpoint) not belonging to them, we can project Π_1 to Π_2 from S. If A, B, C, D are points lying along a straight line in the plane Π_1, and their images are A', B', C', D', respectively, after projection from the point S to Π_2, the following statement holds, as already mentioned before:

$$\frac{\overline{AC}}{\overline{CB}} \div \frac{\overline{AD}}{\overline{DB}} = \frac{\overline{A'C'}}{\overline{C'B'}} \div \frac{\overline{A'D'}}{\overline{D'B'}}.$$

A theorem by Pappus leads to the proof, and also to a deeper understanding of the above statement.



Using the above image, define

$$(abcd) = \frac{\sin ac}{\sin cb} \div \frac{\sin ad}{\sin db}$$

and

$$(ABCD) = \frac{AC}{CB} \div \frac{AD}{DB},$$

where lower-case letters denote lines, and upper-case letters denote points.

Theorem (Pappus): With the above notations,

$$(abcd) = (ABCD).$$

Remark:

$$(abcd) = (ABCD) = (a'b'c'd') = (A'B'C'D').$$

We omit the proof of Pappus' theorem here, but this shows you an example for a theorem belonging to the projective geometry. (Examine the image again!)

Chapter 4

Analytic description of congruent, affine and projective mappings of the plane

Coordinate systems were already used in the first chapter. By convention, we use $\underline{i}$ and $\underline{j}$ as a basis of the plane, they are unit length vectors, perpendicular to each other. The location vector of a point can be constructed as a linear combination of $\underline{i}$ and $\underline{j}$, the coefficients of $\underline{i}$ and $\underline{j}$ will be considered as the coordinates of the given point (the location vector $\underline{p} = x\underline{i} + y\underline{j}$ corresponds to the point (x, y), or $\begin{bmatrix} x & y \end{bmatrix}$ in a matrix form). Having a figure consisting of multiple points, we can write down the coordinates of the points in a matrix, each line corresponding to a corner point in the figure.

Then, many transformations described in the previous chapter can be represented by the image of the vectors $\underline{i}$ and $\underline{j}$: all those affine transformations where the origin is mapped to itself. They are *linear* transformations in the sense that the image of the whole plane is determined by the image of the basis vectors: $\underline{i'}$ and $\underline{j'}$. Then, if a vector is expressed by the basis vectors as $x\underline{i} + y\underline{j}$, then its image will be $x\underline{i'} + y\underline{j'}$. With other words, these transformations preserve the linear combination.

Now put the coordinates of $\underline{i'}$ and $\underline{j'}$ (the image of $\underline{i}$ and $\underline{j}$) to a matrix: both expressed on the original basis (formed by $\underline{i}$ and $\underline{j}$). Let's denote the coordinates of $\underline{i'}$ by a and b, they constitute the first row of the matrix; and the coordinates of $\underline{j'}$ (denote them by c and d) constitute the second row of the matrix.

Proposition: If you multiply the matrix **A** consisting of the coordinates of a given figure (on the left) by the matrix **T** consisting of the coordinates of the image of $\underline{i}$ and $\underline{j}$ (on the right), the resulting product matrix will contain the coordinates of the image of the figure:

$$\begin{array}{rccl} & \mathbf{T} = & \begin{bmatrix} a & b \\ c & d \end{bmatrix} & \\ \mathbf{A} = & \begin{bmatrix} x & y \end{bmatrix} & \begin{bmatrix} ax+cy & bx+dy \end{bmatrix} & = \mathbf{AT} = \begin{bmatrix} x' & y' \end{bmatrix}. \end{array}$$

Proof: The point $\begin{bmatrix} 1 & 0 \end{bmatrix}$ corresponding to $\underline{i}$ is mapped to $\begin{bmatrix} a & b \end{bmatrix}$, and the point $\begin{bmatrix} 0 & 1 \end{bmatrix}$ corresponding to $\underline{j}$ is mapped to $\begin{bmatrix} c & d \end{bmatrix}$. The above described linearity property of the transformation is fulfilled:

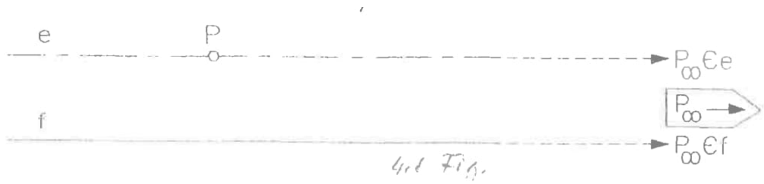
$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} ax+cy & bx+dy \end{bmatrix} = x \begin{bmatrix} a & b \end{bmatrix} + y \begin{bmatrix} c & d \end{bmatrix} = x\underline{i'} + y\underline{j'}.$$

But the set of transformations that can be described this way is restricted. One restriction is that the origin must stay where it was, so even a translation of the figures within the plane is impossible to be represented this way. The other restriction is that parallel lines remain parallel using matrix multiplication in this manner, so projective transformations (central projections) need a more refined approach. We will expand on these topics later. Let us introduce the system of homogenous coordinates in the following section, that will enable us to describe the effect of central projections, too. (The central projection will be built in to the construction of the coordinate system! Then - with these newly defined coordinates - we can use matrix multiplication to represent all the transformations we need.)

4.1 Homogenous coordinates of points in the plane

We describe the points of the Euclidean plane in the Descartes coordinate system using ordered pairs of numbers. If we expand the system of the points of the Euclidean plane with points in infinity - or otherwise, the ideal elements of the plane - we get the concept of the *projective* plane. The question is, how can we assign coordinates to the points of the projective plane (either they are points at a finite or at an infinite distance). How can we define a coordinate system containing them? What will be the rule for assigning coordinates to individual points?

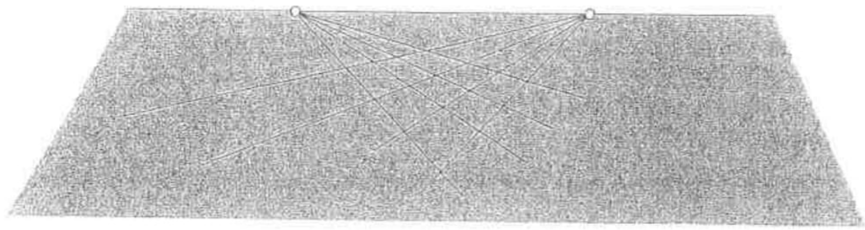
Two straight lines in the Euclidean plane can either be parallel, or they intersect each other. This double nature can be dissolved by adding an ideal element to the set of points of a straight line (it will be called its infinitely far point). The expansion of the plane can be done by adding the same ideal element to all lines that are parallel to each other, - so they will intersect each other in that point. After extending the plane with these ideal elements (infinitely far points), we can say that any two lines intersect each other in this extended plane. (Any straight line contains exactly one ideal point which is the same as the ideal point of any parallel line.)



Consider a Descartes-type coordinate system with 2 perpendicular axes (x and y) in a plane denoted by Π (this will be our projective plane). Then place the plane Π in a 3-dimensional coordinate system with axes x_1, x_2 and x_3, so that the origin of the plane Π coincides with the point $(0, 0, 1)$ (along the x_3 axis of the spacial coordinate system at $x_3 = 1$), and let the x axis in the plane Π be parallel to the x_1 axis in the spacial coordinate system and the y axis be parallel to the x_2 axis, both having the same direction.
That means Π will be identical to the $x_3 = 1$ plane of the spacial coordinate system.
The coordinates of any point of the plane Π given in the spatial coordinate system will be the homogenous coordinates of the point. Simply put, you just add a third coordinate $x_3 = 1$ to the usual two coordinates (considering "normal" points), but it will make more sense: using three coordinate will enable us to describe all the projective transformations (including the remaining affine transformations).
We can think about any point in the space projected to the plane Π using a straight line going through the origin. We can identify all the points along the straight line with the point where the line intersects Π. Similarly, consider how we could connect a newly added ideal point to the origin using a straight line: it does not matter which line we use among parallel lines (all contain the same ideal point), but the parallel lines in the plane Π, that define such a point, don't go through the origin; so we have to take a line pointing to the same direction in the $x_3 = 0$ plane. This is why the third coordinate of the newly added ideal points (or "infinitely far points") will always be 0, not 1.

A plane extended by the ideal points is called a projective plane.

A well-known version of the projective plane is what we use at perspective imaging (see image 4.2). In the image, parallel lines meet at a point (that is called an infinitely far point), and the system of parallel lines in different directions meet in different infinitely far points. All the infinitely far points appear on one line. The "horizon line" can be considered as the locus of infinitely far points, that is, the infinitely far line.

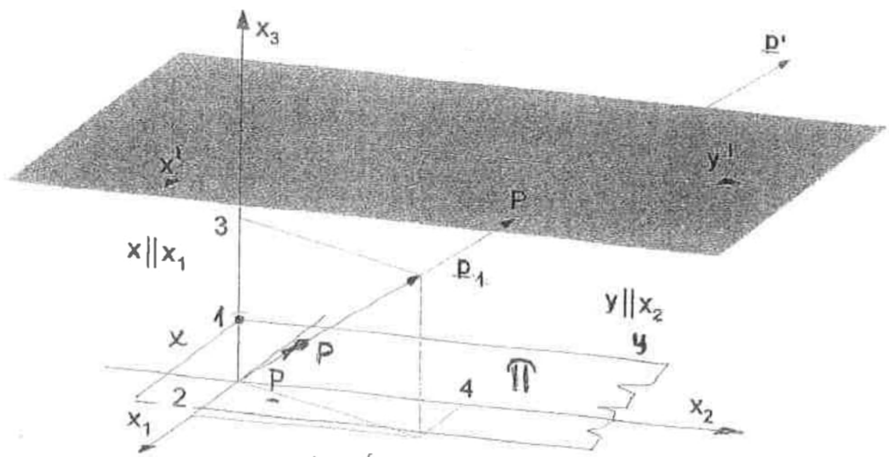


4.2. Fig

It can be proven that the locus of infinitely far points is also a straight line. (We will see projective transformations mapping it to a straight line and mapping an other straight line to the infinitely far line.)

Consider the plane extended by the ideal elements (points and line) – the projective plane. We are going to use a coordinate system that allows us to handle the ideal elements as well.

Let P_1 be a point in the space with location vector $\underline{p_1} = [2\ 4\ 3]$. Then the point P_2 with location vector $\underline{p_2} = [4\ 8\ 6]$ and P_3 with location vector $\underline{p_3} = [-2\ -4\ -3]$ are also mapped to the same point P on the projective plane, using a straight line connecting them with the origin that is parallel with their location vectors. This is a projection from the origin to the plane Π, and we will identify all the points along the same line with one point on the projective plane: $P = \left[\frac{2}{3}\ \frac{4}{3}\ 1\right]$ (see image 4.3). So the *homogenous coordinates* of the point P can be any of these triples.



4.3 fig.

Every point in the plane Π will be described by a class of triples as stated. All the triples corresponding to a point P are coordinates of points in the space along a straight line connecting the origin and the point P. The location vectors of all these points designate the same point of the plane Π.

> The triple [0 0 0] can not be the homogenous coordinate of any point of the plane Π because its location vector does not designate any point (since it has no direction).

Consider an ordinary point of the plane Π (that is not in infinity), then the homogenous coordinates of this point must fulfill the identity

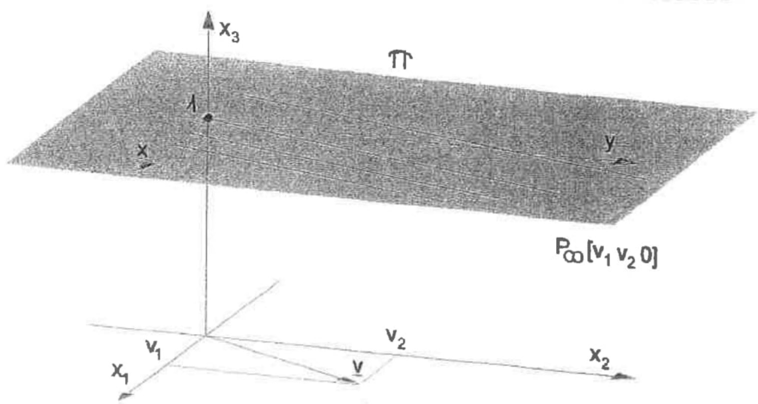
$$x_1 \ : \ x_2 \ : \ x_3 = a_1 \ : \ a_2 \ : \ a_3 = x \ : \ y \ : \ 1$$

that means among the homogenous coordinates of a point you can find the spacial coordinates of the point itself (denoted by P in the image 4.3).

Now consider an infinitely far (ideal, not ordinary) point of the plane Π. It is described by the direction vector $\underline{v} = (v_1, v_2, 0)$ in the (x_1, x_2)-plane because all parallel lines contain the same ideal point. So the homogenous coordinates for the ideal points will be

$$[x_1; x_2; x_3] = [v_1; v_2; 0],$$

as seen in the image 4.4.



4.4 fig

Now we have a coordinate system describing points both in finite and infinite distance, in a uniform way, with triples meaning a unique proportion.

Now parallel lines can be considered as a pencil of rays going through an ideal (infinitely far) point.

Knowing the homogenous coordinates $[x_1; x_2; x_3]$ of an ordinary (not infinitely far) point, we can find the inhomogenous (usual, Descartes-type) coordinates $[x; y]$ in the following way:

$$[x_1; x_2; x_3] \Rightarrow \left[\frac{x_1}{x_3}; \frac{x_2}{x_3}; 1\right] \Rightarrow P(x, y),$$

where

$$x = \frac{x_1}{x_3}, \quad y = \frac{x_2}{x_3} \quad \text{and} \quad x_3 \neq 0.$$

The infinitely far (ideal) points can not be described by inhomogenous coordinates because there you need $x_3 = 0$ that you can not divide by.

If a point P is given by its inhomogenous coordinates x and y, then the homogenous coordinates of can be either $P[x; y; 1]$ or $P[\lambda x; \lambda y; \lambda]$ where $\lambda \in \mathbb{R}$ and $\lambda \neq 0$.

Based on the above discussion, we have two choices to describe points in the plane:

- either we describe their location uniquely by homogenous coordinates in the form $P[x_1; x_2; x_3]$, or
- we can use inhomogenous coordinates in the form $P[x, y]$, but we can not describe the infinitely far (ideal) points this way.

The question is, which one of these two coordinate systems shall we use? − This can not be decided without further information. The system of inhomogenous coordinates is well known and widely used, it can be used effectively in solving many problems. But there are some problems where using homogenous coordinates is not only useful but inevitable.

In the following sections, we will start with the use of normal (inhomogenous) coordinates only, where possible; but we will always write down the equivalent description of the given transformation in the homogenous coordinate system.

4.2 Matrices of transformations in the plane

Points and transformations in the plane

A point in the plane can be described uniquely by its location vector. The coordinates of the location vector are the same as the coordinates of the point itself. The pair of coordinates describing the point can be written as a row vector $\begin{bmatrix} x & y \end{bmatrix}$ or a column vector $\begin{bmatrix} x \\ y \end{bmatrix}$. Figure 4.5 describes an example. (It is more common to use column vectors for the coordinates of a point, but we prefer using *row vectors* because multiplying them by the matrices of transformations will be more convenient. Otherwise these two approaches are equivalent.)

4.5 fig

Consider the following product of matrices:

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ax+cy & bx+dy \end{bmatrix} = \begin{bmatrix} x^* & y^* \end{bmatrix}.$$

The point given by the (inhomogenous) coordinates x and y will be transformed to another point in the plane described by the coordinates x^* and y^* (where $x^* = ax + cy$ and $y^* = bx + dy$). More generally, if A contains the coordinate pairs of the corner points of a figure in the plane, a transformation acting on the figure can be represented by the matrix product

$$A \cdot T = A^*,$$

where T is the matrix of the given transformation.

Any transformation that can be represented by a matrix multiplication using inhomogenous coordinates, maps the midpoint of a line segment AB to the midpoint of the image A^*B^* of the segment. (The proof is based on the linearity property described before.)

Any transformation that can be represented by a matrix multiplication using inhomogenous coordinates, maps a pair of parallel lines to a pair of lines that are parallel, too.

Remark: The equivalent description of these transformations in the homogenous coordinate system is $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

- **The identical and similarity transformations of the plane**

  Let

  $$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

  be the matrix of a transformation.

  1. First, consider the case $a = d = 1$, $b = c = 0$, and denote this transformation by T_I or just simply I:

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} = \begin{bmatrix} x^* & y^* \end{bmatrix},$$

that means the image of the point P is the same as its image P^*.

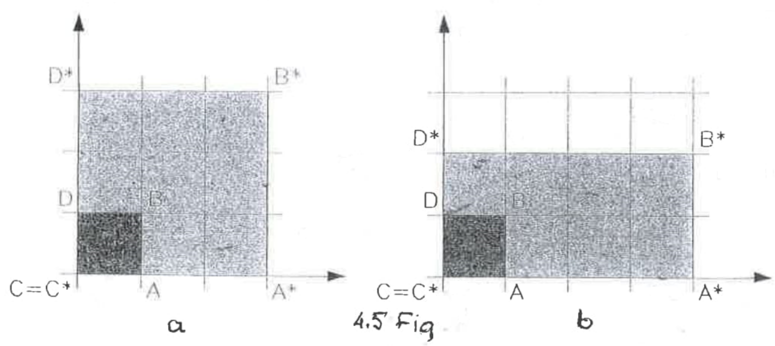
This is called the identical transformation (and identity matrix) because it maps every point to itself.

Remark: The equivalent description of the identical transformation in the homogenous coordinate system is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

2. Consider the case $a = d = s$, $b = c = 0$, and denote this transformation by T_S:

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} = \begin{bmatrix} sx & sy \end{bmatrix} = \begin{bmatrix} x^* & y^* \end{bmatrix}.$$

This is called a similarity transformation, that means a magnification from or shrinking towards the origin, depending on the value of s (see image 4.5 a).



4.5 Fig

Remark: If $a \neq 0$ and $d \neq 0$ and $b = c = 0$, then

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} ax & dy \end{bmatrix} = \begin{bmatrix} x^* & y^* \end{bmatrix}.$$

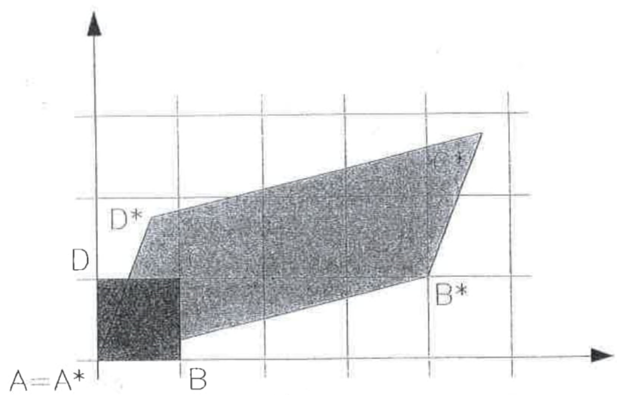
This is a transformation that results in stretching or shrinking along the direction of the x axis, and also along the direction of the y axis, but these effects are not of the same measure if $a \neq d$ (see image 4.5 b).

Remark: The equivalent description of these transformations in the homogenous coordinate system is $\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} a & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

- **Translation within the plane**

  It is not possible to describe translations using a Descartes type coordinate system and matrix multiplication, because the origin has coordinates $(0,0)$, which is not changed using matrix multiplication. E.g. an object with corner points $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$ is transformed in the following way:

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a & b \\ a+c & b+d \\ c & d \end{bmatrix}.$$



4.6 fig

You can see the effect of the transformation in the image 4.6. Apparently, the origin is its own image, substituting any values for a, b, c and d. So multiplication by any matrix $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can not represent a translation.

This is where it becomes necessary to introduce homogenous coordinates:

if we use $\begin{bmatrix} x & y & 1 \end{bmatrix}$ instead of $\begin{bmatrix} x & y \end{bmatrix}$, we can use the matrix

$$T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ m & n \end{bmatrix}$$

to yield

$$\begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ m & n \end{bmatrix} = \begin{bmatrix} x+m & y+n \end{bmatrix} = \begin{bmatrix} x^* & y^* \end{bmatrix},$$

that results in a translation by the vector (m, n) in the original coordinate system.

But we can't change the coordinate system back to using 2 coordinates again, so we need a bigger matrix:

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ m & n & 1 \end{bmatrix}.$$

This will produce the required translation consequently using homogenous coordinates:

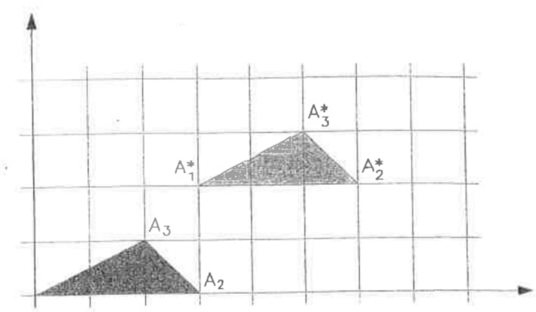
$$\begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ m & n & 1 \end{bmatrix} = \begin{bmatrix} x+m & y+n & 1 \end{bmatrix} = \begin{bmatrix} x^* & y^* & 1 \end{bmatrix}.$$

The translation of the plane by the vector (m, n) can be described as a multiplication by the matrix

$$T_T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ m & n & 1 \end{bmatrix}.$$

Example: Calculate the images A_1^*, A_2^*, A_3^* of the points $A_1(0, 0)$, $A_2(3, 0)$ and $A_3(2, 1)$ after translating them by the vector $\underline{v} = \begin{bmatrix} 3 & 2 \end{bmatrix}$.

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} T = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 6 & 2 & 1 \\ 5 & 3 & 1 \end{bmatrix} = \begin{bmatrix} A_1^* \\ A_2^* \\ A_3^* \end{bmatrix}.$$



4.7. fig

- **Rotation within the plane**

  Rotating the plane about the origin by the angle ϕ maps the point (x, y) to the point with coordinates

$$x^* = x\cos\phi - y\sin\phi$$
$$y^* = x\sin\phi + y\cos\phi$$

  where the angle of rotation is considered positive if we rotate counter-clockwise.

  This statement can be proven by writing down the images of the vectors $\underline{i} = (1, 0)$ and $\underline{j} = (0, 1)$, as described in the beginning of this chapter, and seeing that the matrix of such a rotation is

$$T = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}.$$

  When we are thinking in homogenous coordinates, the transformation takes the following form:

$$\begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} x\cos\phi - y\sin\phi & x\sin\phi + y\cos\phi & 1 \end{bmatrix} = \begin{bmatrix} x^* & y^* & 1 \end{bmatrix}.$$

The rotation of the plane about the origin can be described as a multiplication by the matrix (using homogenous coordinates)

$$T_{Rot} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example: Rotate the square with corner points (2, 0), (2, 4), (4, 4) and (4, 0) about the origin by 90° (see image 4.8).

$$\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} T = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 4 & 1 \\ 4 & 4 & 1 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 90° & \sin 90° & 0 \\ -\sin 90° & \cos 90° & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 2 & 0 & 1 \\ 2 & 4 & 1 \\ 4 & 4 & 1 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 2 & 1 \\ -2 & 4 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} A^* \\ B^* \\ C^* \\ D^* \end{bmatrix}.$$

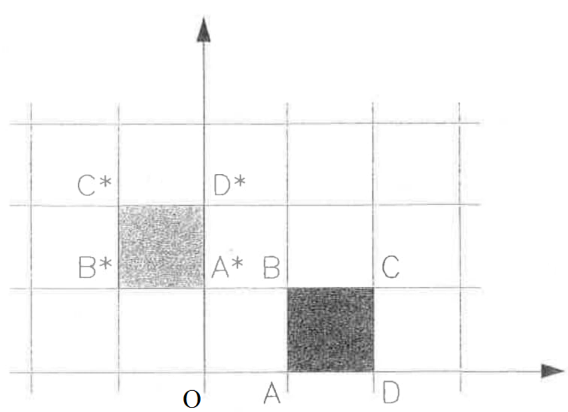


Fig 4.8

The rotation T_{Rot} does not change the orientation of a figure.

We are also able to describe a rotation about an arbitrary point, so that we first translate the point (and the plane with it) in the origin, apply a rotation about the origin, then translate the origin (and the plane with it) back to the original point (that is the inverse of the original translation). So the matrix of such a transformation is

$$T_T T_{Rot} T_T^{-1}.$$

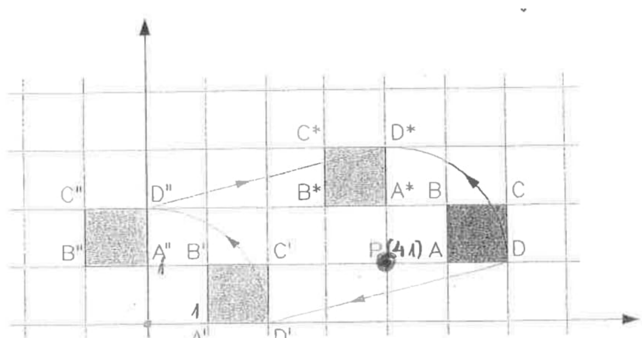
Example: Describe the rotation by 90° about the point $P(4,1)$ (see image 4.9).

Denote the translation of the point $P(4,1)$ to the origin by T_T, the rotation by 90° about the origin by T_{Rot} and the translation back to the origin by T_T^{-1}.

$$\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} T = \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} T_T T_{Rot} T_T^{-1} =$$

$$= \begin{bmatrix} 5 & 1 & 1 \\ 6 & 1 & 1 \\ 6 & 2 & 1 \\ 5 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 1 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 5 & 1 & 1 \\ 6 & 1 & 1 \\ 6 & 2 & 1 \\ 5 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 1 \\ 4 & 3 & 1 \\ 3 & 3 & 1 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} A^* \\ B^* \\ C^* \\ D^* \end{bmatrix}.$$



4.9. fig

- **The matrix of the reflection of the plane through the x or y coordinate axis**

The effect of the transformation $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is that it changes the sign of the second coordinate of a point to the opposite:

$$\begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} x & -y & 1 \end{bmatrix} = \begin{bmatrix} x^* & y^* & 1 \end{bmatrix},$$

that is the same as a reflection of the plane through the x coordinate axis.

The matrix T_x is a reflection of the plane through the x coordinate axis:

$$T_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

the matrix T_y is a reflection of the plane through the y coordinate axis:

$$T_y = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Remark: The equivalent description of these transformations in the normal (inhomogenous) coordinate system is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

A reflection through an arbitrary straight line can be gained by translating it to the origin, rotating it to the x axis, doing a reflection through the x axis, then rotating the straight line back, and then translating it back (of course the whole plane moves with the straight line when we do the translations and rotations):

$$T = T_T T_{Rot} T_x T_{Rot}^{-1} T_T^{-1}.$$

The set of matrices T_T, T_{Rot} and T_x generate the group of congruence transformations of the plane.

If we add the matrix $T_S = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $(s \neq 0)$ to the set of matrices T_T, T_{Rot} and T_x, they generate the group of similarity transformations of the plane.

Remark: With normal (inhomogenous) coordinates, without using the homogenous coordinate system, we are able to describe any rotation about the origin, but not about any other point; also we can describe the reflection through any straight line going through the origin, but not through an arbitrary straight line.

4.3 The affine mappings of the plane to itself

If we add the matrix $T_A = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}$ to the set of matrices T_T, T_{Rot}, T_x and T_S, they generate the group of affine transformations of the plane, where the values a, b, c, d are real numbers such that the rows of the matrix as vectors form an independent system.

Remark: The general form of the elements of the group of affine transformations is $T = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ e & f & 1 \end{bmatrix}$.

The transformation T_A maps the normal (finite) points of the plane to normal (finite) points, too. It maps ideal (infinitely far) points to ideal (infinitely far) points as well.

In order to prove these statements, consider the effect of the transformation T_A on a normal point of the plane:

$$\begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} ax+cy & bx+dy & 1 \end{bmatrix} = \begin{bmatrix} x^* & y^* & 1 \end{bmatrix}.$$

We can see that the transformation T_A maps the point $\begin{bmatrix} x & y & 1 \end{bmatrix}$ to the point $\begin{bmatrix} x^* & y^* & 1 \end{bmatrix}$, consequently all transformations belonging to the affine group map normal points to normal points (using the system of homogenous coordinates).

If we consider the transformation of an infinitely far point

$$\begin{bmatrix} x & y & 0 \end{bmatrix} \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} ax+cy & bx+dy & 0 \end{bmatrix} = \begin{bmatrix} x^* & y^* & 0 \end{bmatrix},$$

we can also conclude that these transformations map ideal points to ideal points.

Image 3.6.b in the previous chapter shows how the sequence of the transformations T_1, T_2, T_3 maps a parallelogram to a square. We consider the square and parallelogram equivalent in the affine sense.

4.4 The projective mappings of the plane to itself

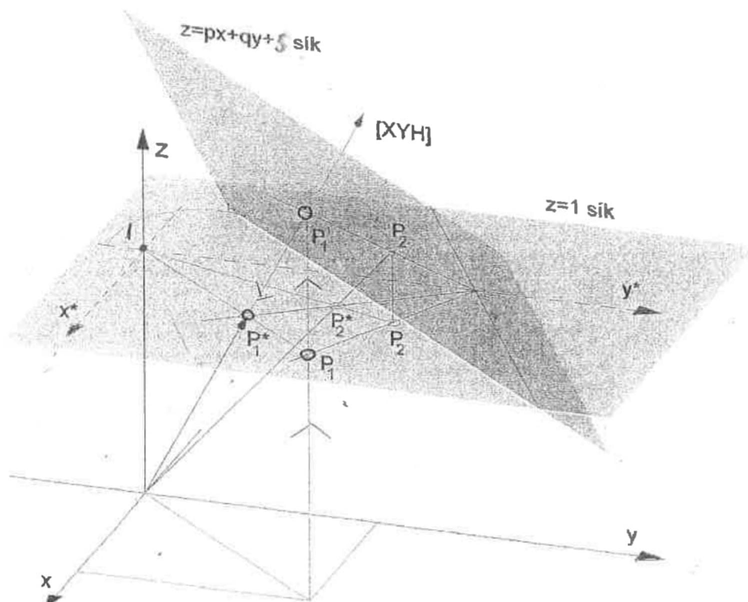
If we add the matrices $T_1 = \begin{bmatrix} 1 & 0 & p \\ 0 & 1 & q \\ 0 & 0 & 1 \end{bmatrix}$ and $T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s \end{bmatrix}$ (where s is not zero) to the group of affine transformation matrices defined earlier, they generate the group of projective transformations of the plane.

Remark: The general form of the matrix of an element of the group of projective transformations is $T = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ where the values $a,\ b,\ c,\ d,\ e,\ f,\ g,\ h,\ i$ are real numbers such that the rows of the matrix as vectors form an independent system.

To understand the effect of the newly added elements, consider the following transformation.

$$\begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & p \\ 0 & 1 & q \\ 0 & 0 & s \end{bmatrix} = \begin{bmatrix} x & y & (pq+qy+s) \end{bmatrix} = \begin{bmatrix} X & Y & 1 \end{bmatrix}$$

where $X = \frac{x}{pq+qy+s}$ and $Y = \frac{y}{pq+qy+s}$. The result of this transformation is the set of points marked by the location vectors $\begin{bmatrix} X & Y & 1 \end{bmatrix}$ in the $z = 1$ plane of the image 4.9b.



4.9. ábra

We can understand the result of the above transformation on the points of the $z = 1$ plane so that we project them to the plane $z = px + qy + s$ using rays parallel to the z axis, and then we project the images of the points back to the $z = 1$ plane using a central projection from / towards the origin. In the image 4.9b we denote these image points by P^*.

In the image 4.9, the plane $z = px + qy + s$ intersects the z axis in s.

If $s = 1$, the $z = px + qy + s$ plane goes through the $z = 1$ point of the z axis, and it has the effect we have just described.

If $p = q = 0$, the transformation $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s \end{bmatrix}$ has a simple geometric meaning:

$$\begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s \end{bmatrix} = \begin{bmatrix} x & y & s \end{bmatrix}$$

where $s \neq 0$, so it can be represented by the homogenous coordinates $\begin{bmatrix} \frac{x}{s} & \frac{y}{s} & 1 \end{bmatrix}$. According to the image 4.10, we project the points of the $z = 1$ plane to the parallel plane $z = s$ using rays parallel to the z axis, and then we project the images of the points back to the $z = 1$ plane using a central projection from / towards the origin (or through, for a negative value of s). The result within the $z = 1$ plane is a magnification (for $|s| < 1$) or shrinking (for $|s| > 1$), starting from or towards the origin (or through, for a negative value of s), same as the result of the transformation $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{s} \end{bmatrix}$.

The group of projective transformations contains, as a special case, the transformations defined earlier: the rotations, reflections and translations, and all affine transformations in general.

According to this transformation group, we can say that two figures in the plane are equivalent even of they are neither congruent nor similar, but there is a sequence of central or parallel projections that maps one into the other.

All the special transformations in the plane are special cases of a matrix of size 3×3 of the form $T = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

Is it true the all these matrices as transformations map a point in the plane to another point in the plane? Consider the transformation represented by the matrix

$$T = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 3 & 4 \\ -3 & -9 & -12 \end{bmatrix}$$

and apply it to the point $T = \begin{bmatrix} 0 & 3 & 1 \end{bmatrix}$:

$$\begin{bmatrix} 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \\ 1 & 3 & 4 \\ -3 & -9 & -12 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}.$$

The transformation T maps the point $P = \begin{bmatrix} 0 & 3 & 1 \end{bmatrix}$ to the "point" $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$. We use a quotation mark because the triple $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ does not represent any point here! In the homogenous coordinate system, we can use any triple to describe a point (finitely or infinitely far) except the triple $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$.

Now it becomes obvious that not all 3×3 matrices correspond to a projective transformation.

If we denote the column vectors of the matrix in the previous example by $\underline{v_1}$, $\underline{v_2}$ and $\underline{v_3}$, we can also write the linear transformation given by the matrix in the following form:

$$\begin{bmatrix} 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix},$$

or equivalently,

$$0 \cdot \underline{v_1} + 3 \cdot \underline{v_2} + 1 \cdot \underline{v_3} = \underline{0}.$$

The last equation means that a linear combination of $\underline{v_1}$, $\underline{v_2}$, $\underline{v_3}$ such that not all the coefficients are zero, yields the $\underline{0}$ vector. According to the definition of independence of vectors, this means $\underline{v_1}$, $\underline{v_2}$ and $\underline{v_3}$ vectors are not independent - they form a dependent system.

> A 3×3 matrix corresponds to a projective transformation of the plane in the homogenous coordinate system if and only if the vectors formed by the columns (or equivalently, the rows) of the matrix are linearly independent vectors.

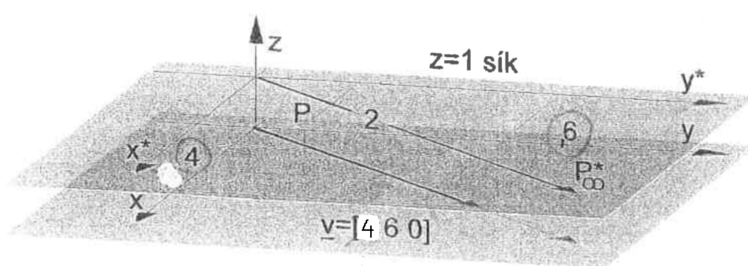
It can be checked that all the 3×3 transformation matrices discussed in this chapter fulfill this criterion, and it can be proven that the product of such matrices also holds this property: its column vectors (and also its row vectors) form an independent system.

An other interesting question is, can a projective transformation (described by a 3×3 matrix) map a point in finite to an infinitely far point? The answer is: yes, it can.

> Projective transformations can only be one-to one mappings of the plane to itself if we include the ideal (infinitely far) points of the plane.

Consider a projective transformation of the plane as an example, and find the image of one of its points in finite:

$$\begin{bmatrix} 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 6 & 0 \end{bmatrix} = \begin{bmatrix} P^* \end{bmatrix}.$$



4.11

The point $\begin{bmatrix} 4 & 6 & 0 \end{bmatrix}$ is nothing else but the infinitely far point of the $z = 1$ plane determined by the direction vector $\underline{v} = \begin{bmatrix} 4 & 6 & 0 \end{bmatrix}$, or the direction $(4, 6)$ within the plane.

In the projective plane, the projective transformations accomplish mappings of the plane to another plane (or itself) so that it is a one-to-one mapping between their points, such that some of the points in finite may be mapped to infinitely far points, and some of the infinitely far points may be mapped to points in finite. Hence in the projective geometry we introduce the infinitely far points, and we abolish their special role by handling them the same way as normal points, not making them exceptional at the description of the transformations of the plane (we mean projective transformations here).

The projective geometry can be gained as a generalization of the Euclidean and Affine geometry, so that the group of congruence, similarity and affine transformations is extended to the group of projective transformations.

Chapter 5

Euclidean, affine and projective transformations in the space

5.1 Some elementary transformations in the space

Similarly to the mapping of a plane to itself, we can also speak about the mappings of the space to the space. Among these transformations, the translations, rotations and reflections of the plane are distinguished.

Using inhomogenous coordinates, a system of n points in the space can be described by an $n \times 3$ matrix. Using homogenous coordinates, we are going to represent it by an $n \times 4$ matrix.

5.1.1 Translation in the space

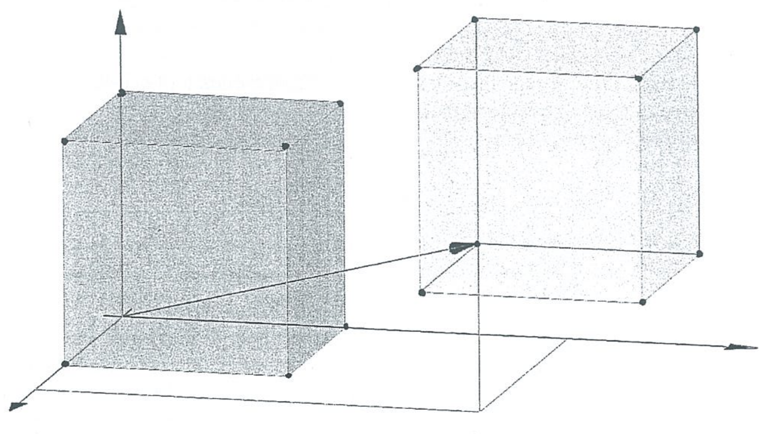
A translation in the space by the vector $\underline{v} = (k, m, n)$ can be described analytically by the following matrix:

$$\begin{bmatrix} x & y & z & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ k & m & n & 1 \end{bmatrix} = \begin{bmatrix} (x+k) & (y+m) & (z+n) & 1 \end{bmatrix}.$$

5.1.2 Rotations about axes in the space

Rotation about a given axis will not change the coordinate belonging to that axis. For example, if we rotate an object about the x axis, then the x coordinate

Figure 5.1: Translation in the space

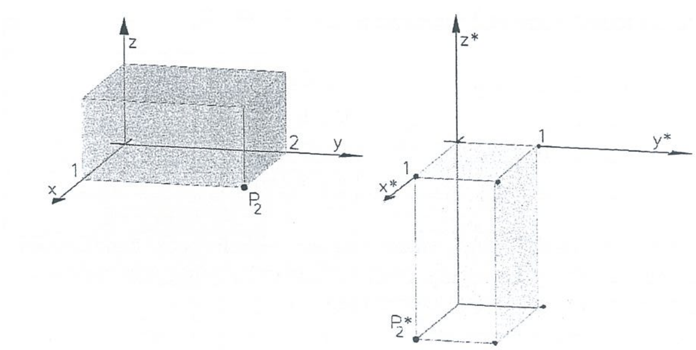


of the points do not change, but the y and z coordinates do. The rotation about the x axis by ϕ is described by the following matrix:

$$T_{Rot(x)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & \sin\phi & 0 \\ 0 & -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The effect of this matrix is a rotation about the x axis so that the positive half of the y axis is turned toward the positive half of the z axis by ϕ.

Figure 5.2: Rotation in the space



Similarly, the matrix

$$T_{Rot(y)}\begin{bmatrix} \cos\phi & 0 & -\sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

describes a rotation about the y axis so that the positive half of the z axis is turned toward the positive half of the x axis by ϕ, and

$$T_{Rot(z)}\begin{bmatrix} \cos\phi & \sin\phi & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

describes a rotation about the z axis so that the positive half of the x axis is turned toward the positive half of the y axis by ϕ.

5.1.3 Reflections through the planes defined by any two of the three axes

Let's add these simple and useful transformations to our set of distinguished transformations.

The matrix

$$T_{xy} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

defines a reflection through the plane defined by the x and y axes, as

$$\begin{bmatrix} x & y & z & 1 \end{bmatrix} T_{xy} = \begin{bmatrix} x & y & -z & 1 \end{bmatrix}.$$

Similarly, the matrices

$$T_{xz} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad T_{yz} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

describe reflections through the planes defined by the pairs of the axes (x, z) and (y, z), respectively.

The description of the Euclidean geometry in the space is determined by the group of similarity transformations in the space. By Euclidean geometry we mean the set of theorems invariant to the elements of the group of similarity transformations. It can be proven that the transformations we have described in this chapter preserve the size of line segments and the size of angles during the transformation of the space (congruence transformations). We also have to add similarity transformations that preserve the *ratio* of line segments and the size of angles during the transformation of the space. These are generated by the transformations defined in the following subsection, added to the group of congruence transformations. For a similarity transformation, s at the first type given in the following subsection can be arbitrary, but the second type of matrix must be restricted to the special case $a = b = c$.

5.1.4 3-dimensional magnification, shrinking, elongation

Consider the following transformation:

$$\begin{bmatrix} x & y & z & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & s \end{bmatrix} = \begin{bmatrix} x & y & z & s \end{bmatrix} \qquad s \neq 0.$$

Its effect is that the point $\begin{bmatrix} x & y & z & 1 \end{bmatrix}$ is transformed to the point $\begin{bmatrix} x & y & z & s \end{bmatrix}$. The inhomogenous coordinates of the image point are

$$x^* = \frac{x}{s}, \quad y^* = \frac{y}{s}, \quad z^* = \frac{z}{s} \quad \text{and} \quad 1.$$

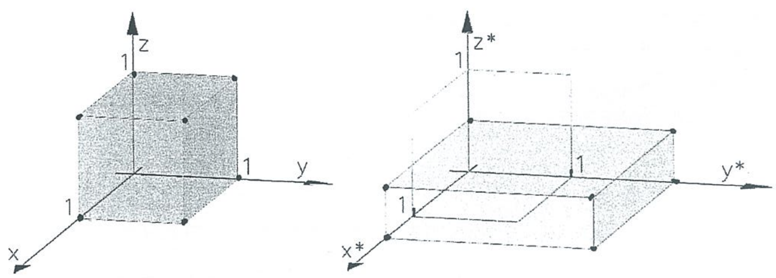
If the transformation matrix has different non-zero elements in the main diagonal (with 1 in the last position), and zeros elsewhere, then the effect of the transformation

$$\begin{bmatrix} x & y & z & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} ax & by & cz & 1 \end{bmatrix} \qquad s \neq 0$$

is a magnification or shrinking in each of the main directions, according to the value of a, b and c being bigger or smaller than 1 in absolute value, and this effect can be different in each of these directions.

Remark: Apart from translations, all these transformations can be described using inhomogenous coordinates and 3×3 matrices.

Figure 5.3: Magnification / shrinking in the space



5.2 Affine transformations in the space

The matrices of the affine transformations have a property in common: the last column of the matrix is $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, if we use homogenous coordinates (that means we have 4×4 matrices in the space). So this group consists of the following matrices:

$$T_A = \begin{bmatrix} a & b & c & 0 \\ d & e & f & 0 \\ g & h & i & 0 \\ k & m & n & 1 \end{bmatrix}.$$

These transformations keep the image of parallel lines parallel, and preserve the ratio of segments of a straight line, e.g. it preserves the midpoint. (But it does not preserve the ratio of segments of different - non-parallel - straight lines.)

Remark: Using inhomogenous coordinates (3 coordinates for the space), all the 3×3 matrices correspond to an affine transformation. There is no restriction for the determinant in this case. (But translations and ideal - infinitely far - points can not be described in this system.)

5.3 Projective transformations in the space

The most general transformations we can describe by matrices are the projective transformations:

$$T_P = \begin{bmatrix} a & b & c & 0 \\ d & e & f & q \\ g & h & i & r \\ k & m & n & s \end{bmatrix}.$$

They have no restriction apart from having a non-zero determinant.

If the rows of the matrix T_P, as vectors, form a linearly independent system then the matrix T_P corresponds to a projective mapping of the space to itself (or to another space).
Equivalent conditions for the matrix of a projective transformation:

- the rows of the matrix T_P, as vectors, form a linearly independent system;
- the columns of the matrix T_P form a linearly independent system;
- the determinant of the matrix is not zero.

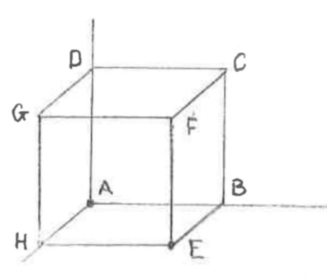
First example for the projective transformation of the plane:
Consider a cube and find its projective image using a transformation with matrix

$$T_P = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & -1 & 2 \\ 2 & -1 & 1 & 1 \end{bmatrix}.$$

The inhomogenous coordinates of the vertices of the cube:

$A:$ (0, 0, 0)
$B:$ (0, 1, 0)
$C:$ (0, 1, 1)
$D:$ (0, 0, 1)

$E:$ (1, 1, 0)
$F:$ (1, 1, 1)
$G:$ (1, 0, 1)
$H:$ (1, 0, 0)



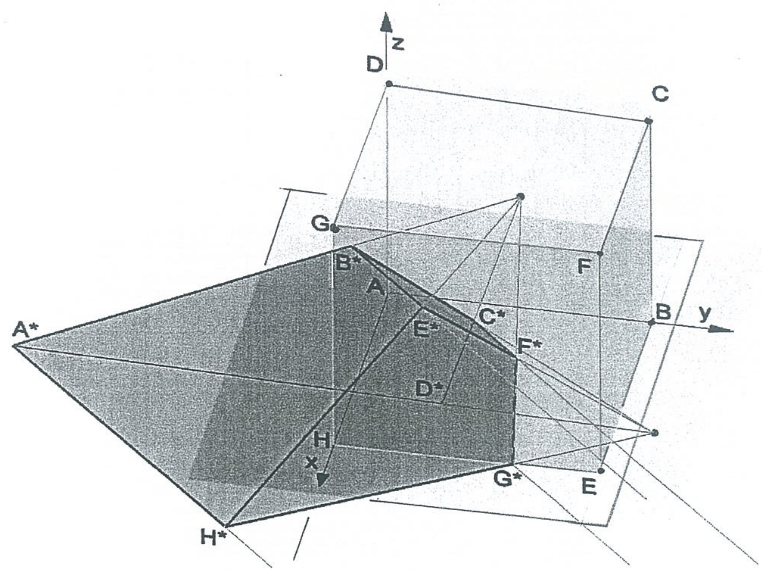
We can only apply the above given transformation if we use homogenous coordinates for the vertices of the cube. Then transformation is

$$T_P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & -1 & 2 \\ 2 & -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 & 1 \\ 2 & 0 & 2 & 3 \\ 2 & 2 & 1 & 5 \\ 2 & 1 & 0 & 3 \\ 3 & 1 & 2 & 3 \\ 3 & 3 & 1 & 5 \\ 3 & 2 & 0 & 3 \\ 3 & 0 & 1 & 1 \end{bmatrix}.$$

That is equivalent to

$$\begin{bmatrix} 2 & -1 & 1 & 1 \\ \frac{2}{3} & 0 & \frac{2}{3} & 1 \\ \frac{2}{5} & \frac{2}{5} & \frac{1}{5} & 1 \\ \frac{2}{3} & \frac{1}{3} & 0 & 1 \\ 1 & \frac{1}{3} & \frac{2}{3} & 1 \\ \frac{3}{5} & \frac{3}{5} & \frac{1}{5} & 1 \\ 1 & \frac{2}{3} & 0 & 1 \\ 3 & 0 & 1 & 1 \end{bmatrix},$$

so you can see the inhomogenous coordinates of the image of the cube in the first three positions of each row. Plotting the image:



As a second example, let us calculate what is the image of the same cube after an other projective transformation.

Let the matrix of the transformation be

$$T_P = \begin{bmatrix} 1 & -2 & -1 & 1 \\ 0 & 2 & 1 & -1 \\ -2 & 1 & 2 & 1 \\ 2 & -1 & 1 & 1 \end{bmatrix}$$

The effect of the transformation on the cube:

$$T_P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -1 & 1 \\ 0 & 2 & 1 & -1 \\ -2 & 1 & 2 & 1 \\ 2 & -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 & 1 \\ 2 & 1 & 2 & 0 \\ 0 & 2 & 4 & 1 \\ 0 & 0 & 3 & 2 \\ 3 & -1 & 1 & 1 \\ 1 & 0 & 3 & 2 \\ 1 & -2 & 2 & 3 \\ 3 & -3 & 0 & 2 \end{bmatrix}.$$

That is equivalent to

$$\begin{bmatrix} 2 & -1 & 1 & 1 \\ 2 & 1 & 2 & 0 \\ 0 & 2 & 4 & 1 \\ 0 & 0 & \frac{3}{2} & 1 \\ 3 & -1 & 1 & 1 \\ \frac{1}{2} & 0 & \frac{3}{2} & 1 \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} & 1 \\ \frac{3}{2} & -\frac{3}{2} & 0 & 1 \end{bmatrix},$$

The inhomogenous coordinates of the vertices of the image of the cube:

$A^*:\ (2,\ -1,\ 1)$

B^* : it can not be described by inhomogenous coordinates. We find the point in the direction of the vector $\underline{v} = (2,\ 1,\ 2)$ in infinity.

$C^*:\ (0,\ 2,\ 4)$

$D^*:\ (0,\ 0,\ \frac{3}{2})$

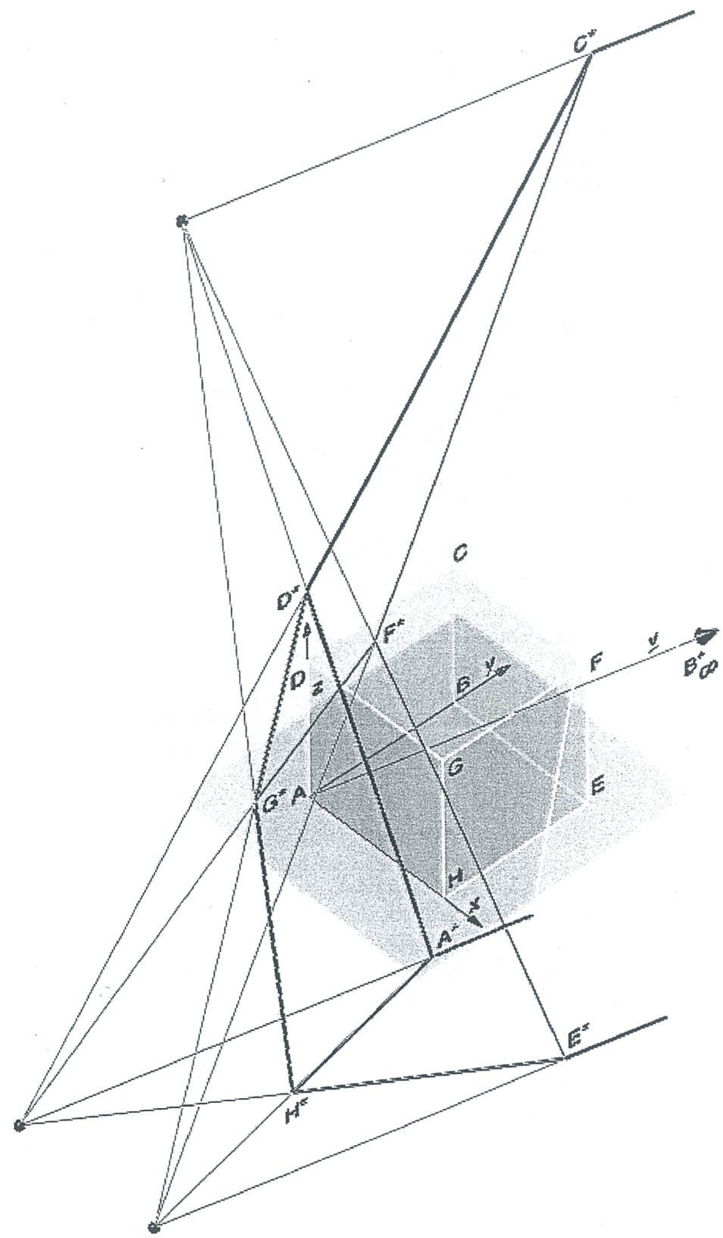
$E^*:\ (3,\ -1,\ 1)$

$F^*:\ (\frac{1}{2},\ 0,\ \frac{3}{2})$

$G^*:\ (\frac{1}{3},\ -\frac{2}{3},\ \frac{2}{3})$

$H^*:\ (\frac{3}{2},\ -\frac{3}{2},\ 0)$

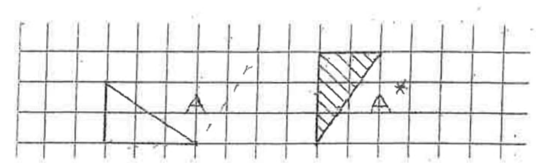
The image of the cube can be seen in the image 5.4.

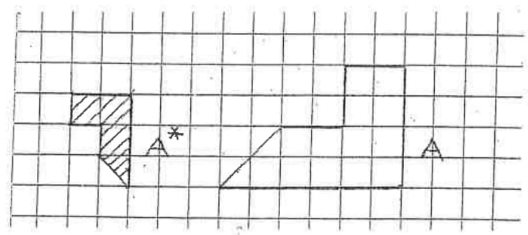

C*
C
D*
F*
D
z
B
y
F
y
G*
A
G
E
H
x
A*
E*
H*

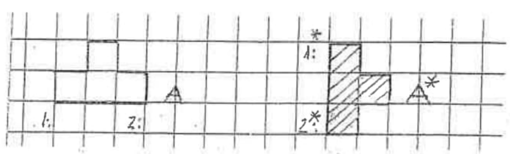
Chapter 6

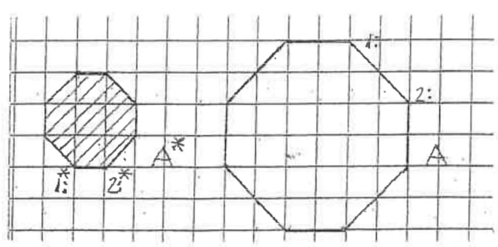
Exercises

1. Let T_T, T_{Rot}, T_{Ref}, T_S mean translation, rotation about the origin, reflection through one of the axes and similarity fixing the origin, respectively. Find what sequence of transformations will map the shape A to A^* and show the phases in the coordinate system.

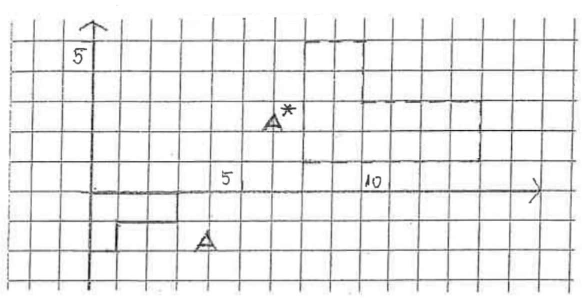






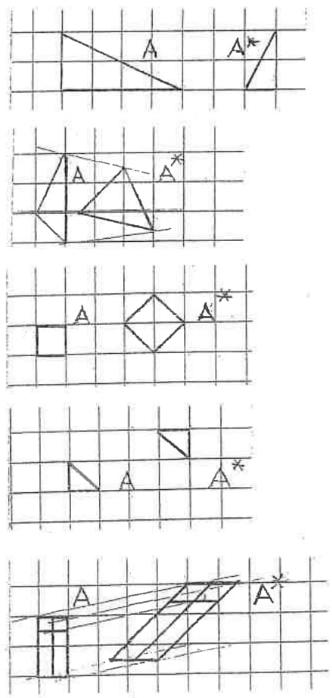


2. Let A and A^* be the shapes seen, and let their matrices also be denoted by A and A^*. Find the matrices T_1, T_2, ... of the above kind (T_T, T_{Rot}, T_{Ref}, T_S) whose product gives a transformation mapping A to A^*, that is, $AT_1T_2... = A^*$.

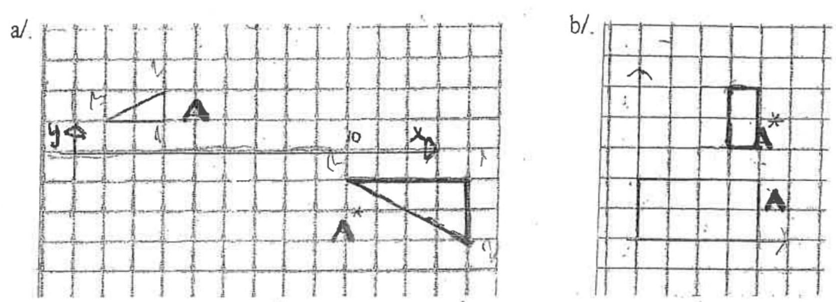


3. In the figures, let A^* be the image of A. Let *1* mean congruent, *2* mean Euclidean (similarity), *3* mean affine, *4* mean projective geometric trans-

formations. Next to the images, write the number(s) of the types of transformations that can move A to A^*.

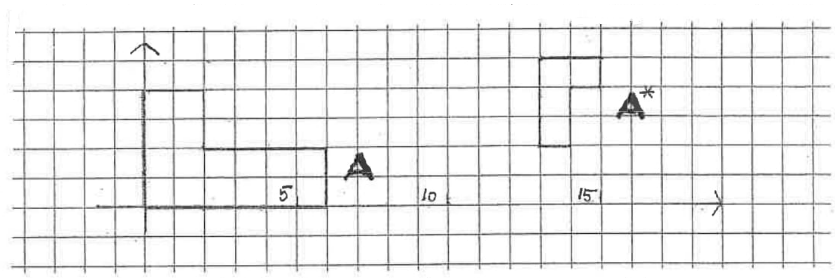


4. Let A and A^* denote the shapes seen on the figure and also their matrices. Find the elements of the matrix T for which $AT = A^*$.



5. Find the matrix product $T = T_1T_2T_3...$ which - multiplying the matrix of a shape A - acts as a rotation by 90 degree about the point $O(4, -2)$.

6. What $T = T_1T_2T_3...$ product, as a transformation matrix, moves A to A^* in the figure?



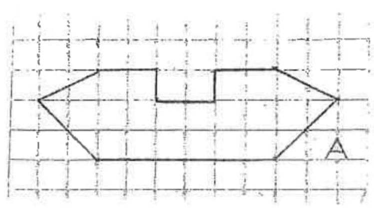
7. Consider the reflection of the plane through the $y = -x + 4$ straight line. What is the matrix of this transformation?

8. Find a matrix product $T = T_1 T_2 ...$ yielding a rotation around an arbitrary point of the plane.

9. What are T_1, $T_2 ...$ matrices if the $T = T_1 T_2 ...$ product is the matrix of the reflection through an arbitrary straight line in the plane.

10. Find the elements of the matrices T_1, T_2, $T_3 ...$ such that the product $T_1 T_2 ... T_n$ shall be the matrix of the reflection through the point $O(0,0)$ in the plane. What product $T_1 T_2 ... T_n$ yields a reflection through an arbitrary point in the plane?

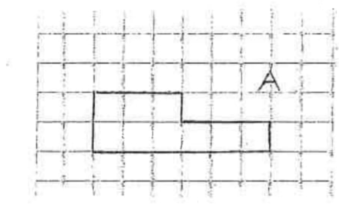
11. Find the matrix of the shape given on the figure (A), then multiply it with the given matrix T. Plot the image $A^* = AT$.



$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 6 & 4 & 1 \end{bmatrix}$$

12. Find the matrix of the shape given on the figure (A), then multiply it with the given matrices: T_1, then T_2, then T_3. Plot the image $A^* = AT_1T_2T_3$.

$$T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad T_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad T_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 6 & -2 & 1 \end{bmatrix}$$



13. Plot the image according to the matrix

$$A = \begin{array}{r} 1: \\ 2: \\ 3: \\ 4: \end{array} \begin{bmatrix} 1 & 1 & 1 \\ 6 & 1 & 1 \\ 6 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix},$$

connecting the points in the order 1 - 2 - 3 - 4 - 1. Calculate the products AT_1, $(AT_1)T_2$, $((AT_1)T_2)T_3$, then plot the images represented by the products, whith

$$T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad T_2 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad T_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

14. Plot the shapes you can get by connecting the points numbered by 1, 2, ..., 6, - the homogeneous coordinates of the points are given in the matrix

$$A = \begin{array}{r} 1: \\ 2: \\ 3: \\ 4: \\ 5: \\ 6: \end{array} \begin{bmatrix} 1 & 1 & 1 \\ 10 & 2 & 2 \\ -5 & -3 & -1 \\ 4 & 3 & 1 \\ 1 & 1 & 0 \\ \frac{1}{3} & 1 & \frac{1}{3} \end{bmatrix}.$$

Plot different shapes according to the following rules connecting the numbered points:

(a) 1 - 2 - 3 - 4 - 1

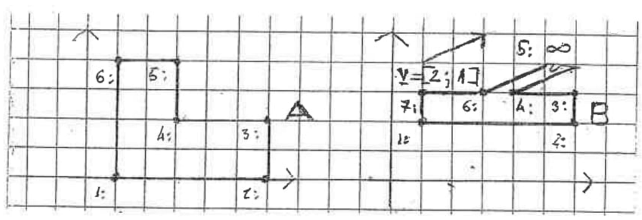
(b) 1 - 2 - 3 - 4

(c) 1 - 2 - 3 - 4 - 5 - 6 - 1

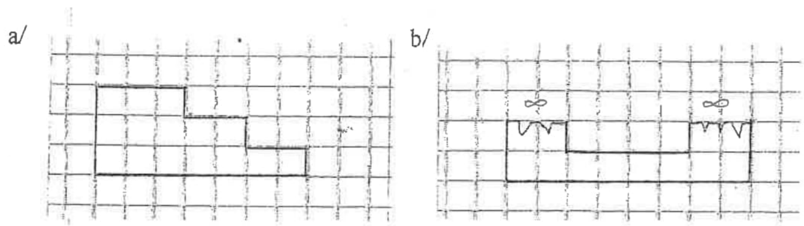
d) 1 - 2 - 3 - 4 - 6 - 1

If the number opening the sequence is the same as the number closing the sequence, then the shape is closed. If the number opening the sequence is not the same as the number closing the sequence, then the shape is open. Connect the points with straight lines.

15. Find the matrix of the following two shapes (A and B), containing the homogeneous coordinates of the points.



16. Find the matrix of the following shapes, containing the homogeneous coordinates of the points.



17. Decide whether the following matrices can or can not be the matrices of projective transformations?

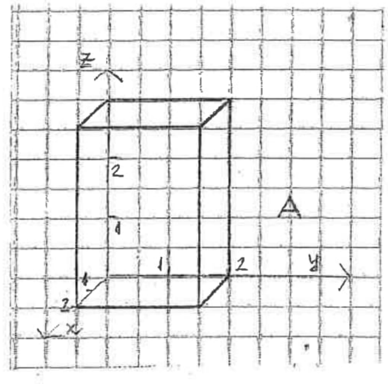
$$T_1 = \begin{bmatrix} 4 & 3 & 0 \\ 2 & 1 & 4 \\ 6 & 4 & 4 \end{bmatrix} \qquad T_2 = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 0 & 0 \\ 4 & 7 & 3 \end{bmatrix} \qquad T_3 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

18. Let A be the object seen in the image (in the space). Plot A^*, the transformed image of A, that we get after the transformation described by the product of T_1, T_2 and T_3.

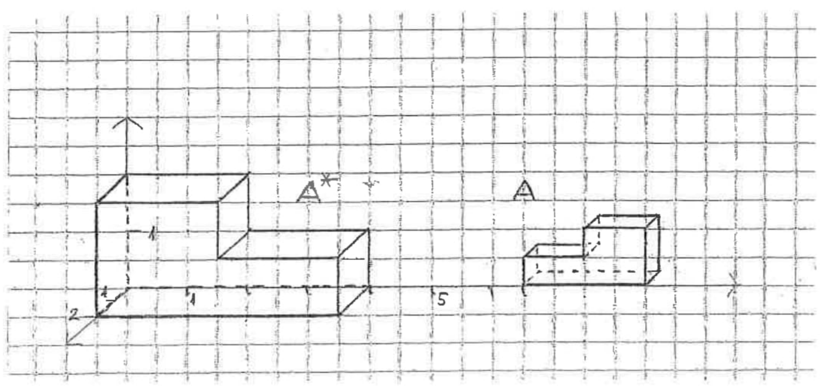
$$T_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{bmatrix}$$

$$T_2 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



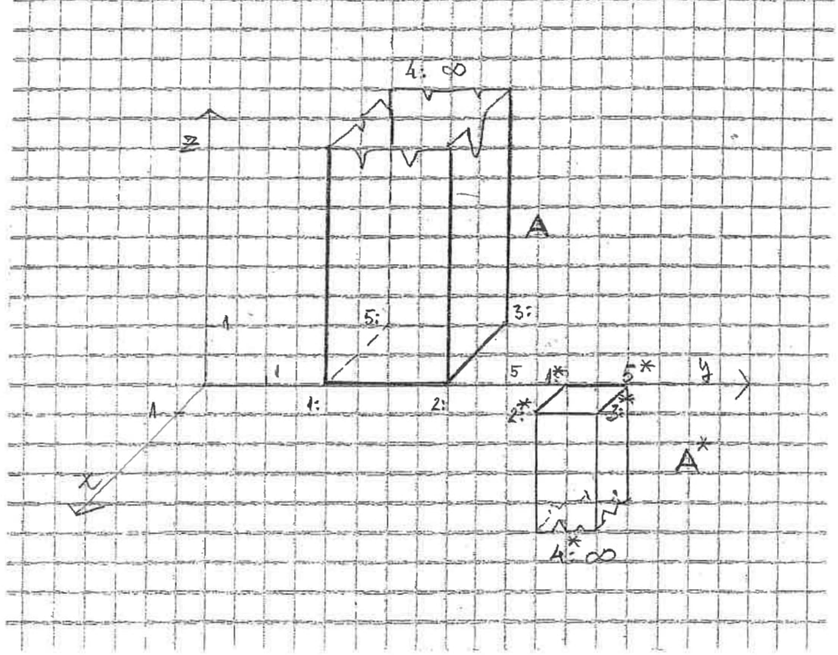
19. What matrices will be the factors of the $T = T_1T_2...T_n$ product, if the transformed image of A is A^*, after the transformation T?



20. Let A be the object in the space that you can see in the image.

    a) Find the matrix consisting of the coordinates of the object A.

    b) What matrices will be the factors of the $T = T_1T_2...T_n$ product, if after the transformation of the space by T, the transformed image of A is A^*?


4: ∞
z
A
3:
5:
1
1
5
1*
5*
y
1
1:
2:
2*
3*
A*
x
4*: ∞